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DETERMINANTS.

BY

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EDITORS' PREFACE.

The volume called Higher Mathematics, the first edition of which was published in 1896, contained eleven chapters by eleven authors, each chapter being independent of the others, but all supposing the reader to have at least a mathematical training equivalent to that given in classical and engineering colleges. The publication of that volume is now discontinued and the chapters are issued in separate form. In these reissues it will generally be found that the monographs are enlarged by additional articles or appendices which either amplify the former presentation or record recent advances. This plan of publication has been arranged in order to meet the demand of teachers and the convenience of classes, but it is also thought that it may prove advantageous to readers in special lines of mathematical literature.

It is the intention of the publishers and editors to add other monographs to the series from time to time, if the call for the same seems to warrant it. Among the topics which are under consideration are those of elliptic functions, the theory of numbers, the group theory, the calculus of variations, and non-Euclidean geometry; possibly also monographs on branches of astronomy, mechanics, and mathematical physics may be included. It is the hope of the editors that this form of publication may tend to promote mathematical study and research over a wider field than that which the former volume has occupied.

December, 1905.

AUTHOR'S PREFACE.

THE author of the present volume feels some embarrassment in having already offered to the public a work upon the Theory of Determinants. The apparently general acceptability of this former work, which has now reached its third edition, doubtless led to his being invited by the editors of Higher Mathematics to prepare for them a chapter upon the same subject. This was done without the least thought of its publication as a separate volume. Now that its issue as such, along with the other chapters, is requested by both the publishers and the editors of Higher Mathematics, it is but just to the author that the above circumstances should be understood lest he be suspected of entertaining an unseemly desire to keep himself before the mathematical public by vain repetition.

The limitations imposed have permitted the addition of only a few articles to the work as originally published; principally those treating of linear substitutions, quantics, invariance, covariance, and functional determinants. Determinants of special forms have not been considered, nor is there the least reference to the application of determinants to geometry. It is hoped, however, that the work may prove useful to the constantly increasing number of students who, while not wishing to specialize in mathematics, desire to obtain the comprehensive view of its methods and processes essential to the successful pursuit of the exact sciences in general.

IOWA CITY, IOWA, U. S. A.,
December, 1905.

CONTENTS.

DETERMINANTS.

ART. 1. INTRODUCTION.

As early as 1693 Leibnitz arrived at some vague notions regarding the functions which we now know as determinants. His researches in this subject, the first account of which is contained in his correspondence with De L'Hospital, resulted simply in the statement of some rather clumsy rules for eliminating the unknowns from systems of linear equations, and exerted no influence whatever upon subsequent investigations in the same direction. It was over half a century later, in 1750, that Gabriel Cramer first formulated an intelligible and general definition of the functions, based upon the recognition of the two classes of permutations, as presently to be set forth.

Though Cramer failed to recognize, even to the same extent as Leibnitz, the importance of the functions thus defined, the development of the subject from this time on has been almost continuous and often rapid. The name "determinant" is due to Gauss, who, with Vandermonde, Lagrange, Cauchy, Jacobi, and others, ranks among the great pioneers in this development.

Within recent years the theory of determinants has come into very general use, and has, in the hands of such mathematicians as Cayley and Sylvester, led to results of the greatest interest and importance, both through the study of special forms of the functions themselves and through their applications.*

* A list of writings on Determinants is given by Muir in Quarterly Journal of Mathematics, 1881, Vol. XVIII, pp. 110–149.

Art. 2. Permutations.

The various orders in which the elements of a group may be arranged in a row are called their permutations.

Any two elements, as a and b, may be arranged in two orders: ab and ba. A third, as c, may be introduced into each of these two permutations in three ways: before either element, or after both; thus giving $3 \times 2 = 6$ permutations of the three elements. In like manner an additional element may be introduced into each of the permutations of i elements in $(i + 1)$ ways: before any one of them, or after all. Hence, in general, if P_i denote the number of permutations of i elements, $P_{i+1} = (i + 1)P_i$. Now, $P_3 = 3 \times 2 \times 1 = 3!$; hence $P_4 = 4 \times 3! = 4!$; and, n being any integer,

$$P_n = n(n - 1)(n - 2) \ldots 1 = n!.$$

That is, the number of permutations of n elements is $n!$.

For all integral values of n greater than unity, $n!$ is an even number.

If the elements of any group be represented by the different letters, $a, b, c, \ldots$, the alphabetical order will be considered as the *natural order* of the elements. If represented by the same letter with different indices, thus:

$$a_1, a_2, a_3, \ldots; \quad \text{or thus:} \quad a', a'', a''', \ldots,$$

the natural order of the elements is that in which the indices form a continually increasing series.

Any two elements, whether adjacent or not, standing in their natural order in a permutation constitute a permanence; standing in an order which is the reverse of the natural, an inversion. Thus, in the permutation *daecb*, the permanences are *de*, *ae*, *ab*, *ac*; the inversions, *da*, *dc*, *db*, *ec*, *eb*, *cb*.

The permutations of the elements of a group are divided into two classes, viz.: even or positive permutations, in which the number of inversions is even; and odd or negative permutations, in which the number of inversions is odd.

When the elements are arranged in the natural order the number of inversions is zero—an even number.

Thus, the even or positive permutations of the elements a_1, a_2, a_3 are

$$a_1\,a_2\,a_3,\quad a_2\,a_3\,a_1,\quad a_3\,a_1\,a_2;$$

while the odd or negative permutations are

$$a_3\,a_2\,a_1,\quad a_1\,a_3\,a_2,\quad a_2\,a_1\,a_3.$$

ART. 3. INTERCHANGE OF TWO ELEMENTS.

It will now be shown that if, in any permutation of the elements of a group, two of the elements be interchanged the class of the permutation will be changed.

Let q and s be the elements in question. Then, representing collectively all the elements which precede these two by P, those which fall between them by R, and those which follow by T, any permutation of the group may be written

$$PqRsT.$$

Of the elements R, supposed to be r in number, let represent

h the number of an order higher than q,
i " " " " " lower " q,
j " " " " " lower " s,
k " " " " " higher " s.

It is evident that no change in the order of the elements qRs can affect their relations to the elements of either P or T. Then, passing from the order $PqRsT$ to the order

$$PRqsT$$

changes the number of inversions by $(h - i)$; and passing from this to the order

$$PsRqT$$

again changes the number of inversions by $(j - k) \pm 1$, the $\left\{\begin{matrix}\text{plus}\\ \text{minus}\end{matrix}\right\}$ sign being used as q is of $\left\{\begin{matrix}\text{lower}\\ \text{higher}\end{matrix}\right\}$ order than s. The total change in the number of inversions due to the interchange of the two elements in question is, therefore,

$$h - i + j - k \pm 1.$$

But since $i = r - h$ and $k = r - j$, this may be written

$$2(h + j - r) \pm 1,$$

which is an odd number for all admissible values of h, j, and r. Hence, the interchange of any two elements in a permutation changes the number of inversions by an odd number, thus changing the class of the permutation.

Art. 4. Positive and Negative Permutations.

Of all the permutations of the elements of a group, one half are even and one half odd.

To prove this, write out all the permutations. Now choose any two of the elements and interchange them in each permutation. The result will be the same set of permutations as before, only differently arranged. But each $\left\{ \begin{matrix} \text{even} \\ \text{odd} \end{matrix} \right\}$ permutation of the old set has been converted into an $\left\{ \begin{matrix} \text{odd} \\ \text{even} \end{matrix} \right\}$ one in the new. Hence, in either set, there are as many even permutations as odd; that is, one half are even and one half odd.

Prob. 1. Classify the following permutations:

(1) $b\,c\,d\,e\,a$; (2) III V I II IV; (3) $k\,n\,i\,m\,l\,j$;
(4) $a''\,a^{\text{v}}\,a'\,a^{\text{iv}}\,a'''$; (5) $\beta\,\epsilon\,\gamma\,\zeta\,\alpha\,\delta$; (6) 5 2 4 1 3;
(7) $x_1\,x_3\,x_0\,x_4\,x_2\,x_5$; (8) F. Tu. M. Th. W.; (9) $\mu\,\kappa\,\nu\,\iota\,\lambda$.

Prob. 2. Derive the formula for the number of permutations of n elements taken m at a time. (Ans. $n!/(n-m)!$.)

Prob. 3. How many combinations of m elements arranged in the natural order may be selected from a group of n elements? (Ans. $n!/m!(n-m)!$.)

Prob. 4. Show that $0! = 1$.

Art. 5. The Determinant Array.

Assume n^2 elements arranged in n vertical ranks or columns, and n horizontal ranks or rows, thus:

$$\begin{matrix} a_1' \; a_1'' \ldots a_1^{(n)} \\ a_2' \; a_2'' \ldots a_2^{(n)} \\ \ldots\ldots\ldots\ldots \\ a_n' \; a_n'' \ldots a_n^{(n)}. \end{matrix}$$

In this array all the elements in the same column have the same superscript, and those in the same row the same subscript. The columns being arranged in order from left to right, and the rows likewise in order from the top row downward, the position of any element of the array is shown at once by its indices. Thus, a_5''' is in the third column and the fifth row of the above array.

The diagonal passing through the elements $a_1', a_2'', \ldots a_n^{(n)}$ is called the principal diagonal of the array; that passing through $a_n', a_{n-1}'', \ldots a_1^{(n)}$, the secondary diagonal. The position occupied by the element a_1' is designated as the leading position.

ART. 6. DETERMINANT AS FUNCTION OF n^2 ELEMENTS.

The array just considered, inclosed between two vertical bars, thus:

$$\begin{vmatrix} a_1' & a_1'' & \ldots & a_1^{(n)} \\ a_2' & a_2'' & \ldots & a_2^{(n)} \\ \ldots & \ldots & \ldots & \ldots \\ a_n' & a_n'' & \ldots & a_n^{(n)} \end{vmatrix}$$

is used in analysis to represent a certain function of its n^2 elements called their determinant.* This function may be defined as follows:

Write down the product of the elements on the principal diagonal, taking them in the natural order; thus:

$$a_1' \, a_2'' \, a_3''' \ldots a_n^{(n)}.$$

This product is called the principal term of the determinant. Now permute the subscripts in this principal term in every possible way, leaving the superscripts undisturbed. To such of the $n!$ resulting terms as involve the even permutations of the subscripts give the positive sign; to those involving the odd

* This notation was first employed by Cauchy in 1815. See Dostor's Théorie des déterminants, Paris, 1877.

permutations, the negative sign. The algebraic sum of all the terms thus formed is the determinant represented by the given array.

ART. 7. EXAMPLES OF DETERMINANTS.

Applying the process above explained to the array of four elements gives

$$\begin{vmatrix} a_1' & a_1'' \\ a_2' & a_2'' \end{vmatrix} \equiv a_1'a_2'' - a_2'a_1''. \tag{1}$$

As an example of a determinant of nine elements, with its expansion, may be written

$$\begin{vmatrix} a_1' & a_1'' & a_1''' \\ a_2' & a_2'' & a_2''' \\ a_3' & a_3'' & a_3''' \end{vmatrix} \equiv \begin{array}{l} + a_1'a_2''a_3''' + a_2'a_3''a_1''' + a_3'a_1''a_2''' \\ - a_3'a_2''a_1''' - a_1'a_3''a_2''' - a_2'a_1''a_3'''. \end{array} \tag{2}$$

It is evident, from the mode of its formation, that each term of the expansion of a determinant contains one, and only one, element from each column and each row of the array.

It follows that every complete determinant is a homogeneous function of its elements. The degree of this function, with respect to its elements, is called the order of the determinant. Thus, (1) and (2) are of the second and third order respectively.

The definition of a determinant given in the preceding article is once more illustrated by the following example of a determinant of the fourth order with its complete development:

$$\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix} \equiv \left. \begin{array}{l} + a_1b_2c_3d_4 - a_1b_2c_4d_3 - a_1b_3c_2d_4 + a_1b_4c_2d_3 \\ + a_1b_3c_4d_2 - a_1b_4c_3d_2 - a_2b_1c_3d_4 + a_2b_1c_4d_3 \\ + a_3b_1c_2d_4 - a_4b_1c_2d_3 - a_3b_1c_4d_2 + a_4b_1c_3d_2 \\ + a_2b_3c_1d_4 - a_2b_4c_1d_3 - a_3b_2c_1d_4 + a_4b_2c_1d_3 \\ + a_3b_4c_1d_2 - a_4b_3c_1d_2 - a_2b_3c_4d_1 + a_2b_4c_3d_1 \\ + a_3b_2c_4d_1 - a_4b_2c_3d_1 - a_3b_4c_2d_1 + a_4b_3c_2d_1 \end{array} \right\} \tag{3}$$

It will be noticed that, in this case, the columns are ranked alphabetically instead of by the numerical values of a series of indices.

ART. 8. NOTATIONS.

Besides the notations already employed, the following is very extensively used :

$$\begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix}.$$

This is called the double-subscript notation ; the first subscript indicating the rank of the row, the second that of the column. Thus the element a_{23} is in the second row and the third column. The letters are sometimes omitted, the elements being thus represented by the double subscripts alone.*

Instead of writing out the array in full, it is customary, when the elements are merely symbolic, to write only the principal term and enclose it between vertical bars. This is called the umbral notation. Thus, the determinant of the nth order is written

$$| a_1' a_2'' \dots a_n^{(n)} | ;$$

or, using double subscripts,

$$| a_{11} a_{22} \dots , a_{nn} | .$$

These last two forms are sometimes still further abridged to

$$| a_1^{(n)} | \quad \text{and} \quad | a_{1,n} | ,$$

respectively.

Prob. 5. Write out the developments of the following determinants:

(1) $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$; (2) $\begin{vmatrix} p' & p'' \\ q' & q'' \end{vmatrix}$; (3) $\begin{vmatrix} p' & q' \\ p'' & q'' \end{vmatrix}$; (4) $\begin{vmatrix} a & b \\ \alpha & \beta \end{vmatrix}$;

* Leibnitz indicated the elements of a determinant in this same manner, though he made no use of the array.

(5) $| a_1 b_2 c_3 |$; (6) $\begin{vmatrix} p' & p'' & p''' \\ q' & q'' & q''' \\ r' & r'' & r''' \end{vmatrix}$; (7) $\begin{vmatrix} p' & q' & r' \\ p'' & q'' & r'' \\ p''' & q''' & r''' \end{vmatrix}$; (8) $\begin{vmatrix} a & b & c \\ \alpha & \beta & \gamma \\ x & y & z \end{vmatrix}$;

(9) $| 11, 22 |$; (10) $| a_{1,3} |$; (11) $| l_0 m_1 n_2 |$; (12) $| a_{11} a_{22} a_{33} a_{44} |$.

Prob. 6. How many terms are there in the development of the determinant $| a_1^{\text{vi}} |$?

In the above determinant tell the signs of the terms:

$$(1)\ a_6' a_2'' a_1''' a_4^{\text{iv}} a_5^{\text{v}} a_3^{\text{vi}}; \qquad (2)\ a_1' a_3'' a_2''' a_5^{\text{iv}} a_6^{\text{v}} a_4^{\text{vi}};$$

$$(3)\ a_6' a_4'' a_5''' a_1^{\text{iv}} a_3^{\text{v}} a_2^{\text{vi}}.$$

Prob. 7. Show that in the expansion of any determinant, all of whose elements are positive, one half the terms are positive and one half negative.

Prob. 8. In determinants of what orders is the term containing the elements on the secondary diagonal (called the secondary term) positive?

Prob. 9. What is the order of the determinant whose secondary term contains 10 inversions? 36 inversions?

Prob. 10. In the expansion of a determinant of the nth order, how many terms contain the leading element?

ART. 9. SECOND AND THIRD ORDERS.

Simple rules will now be given for writing out the expansions of determinants of the second and third orders directly from the arrays by which they are represented.

To expand a determinant of the second order, write the product of the elements on the principal diagonal minus the product of those on the secondary diagonal, thus:

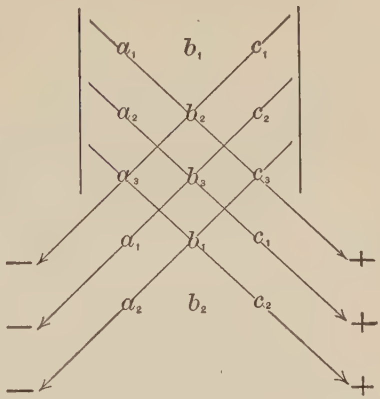
$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} \equiv ad - bc.$$

Likewise, $$\begin{vmatrix} -9 & 5 \\ -2 & \frac{1}{3} \end{vmatrix} = -3 + 10 = 7.$$

The following method is applicable to determinants of the third order:*

* This method was first given by Sarrus, and is often called the rule of Sarrus; see Finck's Éléments d'Algèbre, 1846, p. 95.

Beneath the square array let the first two rows be repeated in order, as shown in the figure. Now write down six terms, each the product of the three elements lying along one of the six oblique lines parallel to the diagonals of the original square. Give to those terms whose elements lie on lines parallel to the principal diagonal the positive sign; to the others, the negative sign. The result is the required expansion. Applying the method to the determinant just written gives



$$|a_1b_2c_3| = a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2 - a_3b_2c_1 - a_1b_3c_2 - a_2b_1c_3.$$

After a little practice the repetition of the first two rows will be dispensed with.

The above methods are especially useful in expanding determinants whose elements are not marked with indices, or in evaluating those having numerical elements. No such simple methods can be given for developing determinants of higher orders, but it will be shown later that these can always be resolved into determinants of the third or second order.

Prob. 11. Develop the following determinants:

(1) $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$; (2) $\begin{vmatrix} 0 & -n & -m \\ n & 0 & -l \\ m & l & 0 \end{vmatrix}$; (3) $\begin{vmatrix} A & c & b \\ c & B & a \\ b & a & C \end{vmatrix}$;

(4) $\begin{vmatrix} x_0 & y_0 & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix}$; (5) $\begin{vmatrix} 1 & P & Q \\ 0 & \cos\alpha & \sin\beta \\ 0 & \sin\alpha & \cos\beta \end{vmatrix}$; (6) $\begin{vmatrix} \cos\alpha & \sin\beta \\ \sin\alpha & \cos\beta \end{vmatrix}$;

(7) $\begin{vmatrix} 1 & \cos\alpha \\ \cos\alpha & 1 \end{vmatrix}$; (8) $\begin{vmatrix} 1 & \sqrt{-1} \\ 4 & \sqrt{-2} \end{vmatrix}$; (9) $\begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix}$.

Prob. 12. Evaluate the following:

(1) $\begin{vmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{vmatrix}$; (2) $\begin{vmatrix} -2 & -2 & \frac{1}{2} \\ 0 & -2 & 0 \\ 12 & 2 & 1 \end{vmatrix}$; (3) $\begin{vmatrix} -1 & -\sqrt{-1} & -\sqrt{-1} \\ \sqrt{-1} & -1 & -\sqrt{-1} \\ \sqrt{-1} & \sqrt{-1} & -1 \end{vmatrix}$.

(Ans. 18; 16; 2.)

Art. 10. Interchange of Rows and Columns.

Any term in the development of the determinant $|a_1^{(n)}|$ may be written

$$\pm a_h' a_i'' a_j''' \ldots a_l^{(n)},$$

in which $hij \ldots l$ is some permutation of the subscripts 1, 2, 3, ... n. Designate by u the number of inversions in $hij \ldots l$. Also, let v be the number of interchanges of two elements necessary to bring the given term into the form

$$\pm a_1^{(p)} a_2^{(q)} a_3^{(r)} \ldots a_n^{(t)},$$

in which the subscripts are arranged in the natural order, while $pqr \ldots t$ is a certain permutation of the superscripts $'$, $''$, $'''$, ... $^{(n)}$.

This permutation is even or odd according as v is even or odd. But u and v are obviously of the same class; that is, both are even or both odd. Hence the permutations $hij \ldots l$ and $pqr \ldots t$ are of the same class; and the term will have the same sign, whether the sign be determined by the class of the permutation of the subscripts when the superscripts stand in the natural order, or by the class of the permutation of the superscripts when the order of the subscripts is natural.

It follows that the same development of the determinant array will be obtained if, instead of proceeding as indicated in Art. 6, the superscripts of the principal term be permuted, the subscripts being left in the natural order, and the sign of each of the resulting terms written in accordance with the class of the permutations of its superscripts.

Passing from one of these methods of development to the other amounts to the same thing as changing each column of the array into a row of the same rank, and *vice versa*. Hence, a determinant is not altered by changing the columns into corresponding rows and the rows into corresponding columns. Thus:

$$\begin{vmatrix} a_1' & a_1'' & \ldots & a_1^{(n)} \\ a_2' & a_2'' & \ldots & a_2^{(n)} \\ \ldots & \ldots & \ldots & \ldots \\ a_n' & a_n'' & \ldots & a_n^{(n)} \end{vmatrix} \equiv \begin{vmatrix} a_1' & a_2' & \ldots & a_n' \\ a_1'' & a_2'' & \ldots & a_n'' \\ \ldots & \ldots & \ldots & \ldots \\ a_1^{(n)} & a_2^{(n)} & \ldots & a_n^{(n)} \end{vmatrix}.$$

Whatever theorem, therefore, is demonstrated with reference to the rows of a determinant is also true with reference to the columns.

The rows and columns of a determinant array are alike called lines.

Art. 11. Interchange of Two Parallel Lines.

If any two parallel lines of a determinant be interchanged, the determinant will be changed only in sign.

For, interchanging any two parallel lines of a determinant array amounts to the same thing as interchanging, in every term of the expansion, the indices which correspond to these lines. Since this changes the class of each permutation of the indices in question from odd to even or from even to odd, it changes the sign of each term of the expansion, and therefore that of the whole determinant.

It follows from the above that if any line of a determinant be passed over m parallel lines to a new position in the array the new determinant will be equal to the original one multiplied by $(-1)^m$.

The element $a_k^{(s)}$ may be brought to the leading position by passing the kth row over the $(k-1)$ preceding rows, and the sth column over the $(s-1)$ preceding columns. This being done the determinant is multiplied by

$$(-1)^{k-1} \cdot (-1)^{s-1} = (-1)^{k+s},$$

which changes its sign or not according as $(k+s)$ is odd or even.

The position occupied by $a_k^{(s)}$ is called a positive position when $(k+s)$ is even; a negative position when $(k+s)$ is odd.

Art. 12. Two Identical Parallel Lines.

A determinant in which any two parallel lines are identical is equal to zero.

For the interchange of these two parallel lines, while it

changes the sign of the determinant, will in no way alter its value. The value then, if finite, can only be zero.

ART. 13. MULTIPLYING BY A FACTOR.

Multiplying each element of a line of a determinant by a given factor multiplies the determinant by that factor.

Since each term of the development contains one and only one element from the line in question (Art. 7), then multiplying each element of this line by the given factor multiplies each term of the development, and therefore the whole determinant, by the same factor.

It follows that, if the elements of any line of a determinant contain a common factor, this factor may be canceled and written outside the array as a factor of the whole determinant; thus:

$$\begin{vmatrix} a_{11} \ldots m\, a_{1i} \ldots a_{1n} \\ a_{21} \ldots m\, a_{2i} \ldots a_{2n} \\ \ldots\ldots\ldots\ldots \\ a_{n1} \ldots m\, a_{ni} \ldots a_{nn} \end{vmatrix} = m \, | \, a_{11}\, a_{22} \ldots a_{nn} \, | \, .$$

A determinant in which the elements of any line have a common ratio to the corresponding elements of any parallel line is equal to zero. For this common ratio may be written outside the array, which will then have two identical lines. Its value is therefore zero (Art. 12).

A determinant having a line of zeros is equal to zero.

ART. 14. A LINE OF POLYNOMIAL ELEMENTS.

A determinant having a line of elements each of which is the sum of two or more quantities can be expressed as the sum of two or more determinants.

Let
$$\begin{vmatrix} a_1 & (b_1 - b_1' + b_1'' \pm \ldots) & c_1 \ldots \\ a_2 & (b_2 - b_2' + b_2'' \pm \ldots) & c_2 \ldots \\ a_3 & (b_3 - b_3' + b_3'' \pm \ldots) & c_3 \ldots \\ \ldots & \ldots\ldots\ldots\ldots & \ldots \end{vmatrix} \equiv \varDelta \qquad (1)$$

be such a determinant. Then, if

$$B_i \equiv b_i - b_i' + b_i'' \pm \ldots,$$

any term of the expansion of the determinant $\varDelta$ is

$$\pm a_h B_i c_j \dots = \pm a_h b_i c_j \dots \mp a_h b_i' c_j \dots$$
$$\pm a_h b_i'' c_j \dots \pm \dots \qquad (2)$$

The terms in the expansion of $\varDelta$ are obtained by permuting the subscripts $h, i, j, \dots$ of $a_h B_i c_j \dots$. But permuting at the same time the subscripts of the terms in the second member of (2), and giving to each term thus obtained its proper sign, there results

$$\varDelta \equiv |a_1 B_2 c_3 \dots| = |a_1 b_2 c_3 \dots| - |a_1 b_2' c_3 \dots| + |a_1 b_2'' c_3 \dots| \pm \dots,$$

which proves the theorem.

Art. 15. Composition of Parallel Lines.

If each element of a line of a determinant be multiplied by a given factor and the product added to the corresponding element of any parallel line, the value of the determinant will not be changed ; thus:

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ \dots & \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nn} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & (a_{13} + ma_{11}) & \dots & a_{1n} \\ \dots & \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & (a_{n3} + ma_{n1}) & \dots & a_{nn} \end{vmatrix}.$$

This will appear upon resolving the second member into two determinants (Art. 14), one of which will be the given determinant, while the other, upon removal of the given factor, will vanish because of having two identical lines.

In like manner any number of parallel lines may be combined without changing the value of the determinant, care being taken not to modify in any way the elements to which are added multiples of corresponding elements from other parallel lines. For example, $|a_{1,n}|$ is equivalent to

$$\begin{vmatrix} a_{11} & (la_{11} + a_{12} - ma_{13} + \dots) & a_{13} & \dots & a_{1n} \\ (a_{21} + \lambda a_{11}) & \begin{matrix}(l(a_{21} + \lambda a_{11}) + (a_{22} + \lambda a_{12}) \\ -m(a_{23} + \lambda a_{13}) + \dots)\end{matrix} & (a_{23} + \lambda a_{13}) & \dots & (a_{2n} + \lambda a_{1n}) \\ \dots & \dots & \dots & \dots & \dots \\ a_{n1} & (la_{n1} + a_{n2} - ma_{n3} + \dots) & a_{n3} & \dots & a_{nn} \end{vmatrix}.$$

ART. 16. BINOMIAL FACTORS.

A determinant which is a rational integral function of a and of b, such that if b is substituted for a the determinant vanishes, contains $(a - b)$ as a factor. For example,

$$\Delta \equiv \begin{vmatrix} a^2 - p^2 & a - q & a + r \\ b^2 - p^2 & b - q & b + r \\ p & q & r \end{vmatrix}$$

is divisible by $(a - b)$.

To prove this, let the expansion of any such determinant be written in the form

$$\Delta = m_0 + m_1 a + m_2 a^2 + \ldots,$$

the coefficients $m_0, m_1, m_2, \ldots$ being independent of a. Now when b is substituted for a the determinant vanishes. Hence,

$$0 = m_0 + m_1 b + m_2 b^2 + \ldots$$

Subtracting this from the preceding gives

$$\Delta = m_1(a - b) + m_2(a^2 - b^2) + \ldots$$

This being divisible by $(a - b)$, the theorem is proven.

Prob. 13. Prove the following without expansion :

(1) $\begin{vmatrix} 0 & -x & x \\ my & 0 & -y \\ -mnz & nz & 0 \end{vmatrix} = 0;$ (2) $\begin{vmatrix} 0 & c - b \\ -c & 0 & a \\ b & -a & 0 \end{vmatrix} = 0;$

(3) $\begin{vmatrix} b + c & a & a \\ b & c + a & b \\ c & c & a + b \end{vmatrix} = 2 \begin{vmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{vmatrix};$

(4) $\begin{vmatrix} \frac{b^2 + c^2}{a} & a & a \\ b & \frac{c^2 + a^2}{b} & b \\ c & c & \frac{a^2 + b^2}{c} \end{vmatrix} = 2 \begin{vmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{vmatrix};$

(5) $\begin{vmatrix} a & \sin A & b - c \\ b & \sin B & c - a \\ c & \sin C & a - b \end{vmatrix} = 0$, the elements referring to the triangle ABC.

Prob. 14. Prove that

$$\begin{vmatrix} 1 & x-a & y-b \\ 1 & x_1-a & y_1-b \\ 1 & x_2-a & y_2-b \end{vmatrix} = \begin{vmatrix} 1 & x & y \\ 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \end{vmatrix} = \begin{vmatrix} 1 & x & y \\ 0 & x_1-x & y_1-y \\ 0 & x_2-x & y_2-y \end{vmatrix}.$$

Prob. 15. Find the value of θ in the equation

$$\begin{vmatrix} \sin\theta & \sin\theta & 0 \\ 1 & 0 & 1 \\ 0 & \cos\theta & \cos\theta \end{vmatrix} = 0. \quad (\text{Ans. } \theta = \cancel{\pi/4}.) \ n\pi/2$$

Prob. 16. Show that the proportion $a:b::l:m$ may be written in the form $\begin{vmatrix} a & b \\ l & m \end{vmatrix} = 0$; and from the properties of this determinant prove the common theorems in proportion.

Prob. 17. Show that the determinant $\begin{vmatrix} ab & c^2 & c^2 \\ a^2 & bc & a^2 \\ b^2 & b^2 & ca \end{vmatrix}$ contains the factor $(bc + ca + ab)$.

Prob. 18. Resolve the following determinants into factors:*

(1) $\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$; (2) $\begin{vmatrix} 1 & a & a^2 & a^3 \\ 1 & b & b^2 & b^3 \\ 1 & c & c^2 & c^3 \\ 1 & d & d^2 & d^3 \end{vmatrix}$; (3) $\begin{vmatrix} 1 & a_1 & a_1^2 & \dots & a_1^{n-1} \\ 1 & a_2 & a_2^2 & \dots & a_2^{n-1} \\ \dots & \dots & \dots & \dots & \dots \\ 1 & a_n & a_n^2 & \dots & a_n^{n-1} \end{vmatrix}$;

(4) $\begin{vmatrix} 1 & 1 & 1 & 1 \\ a & b & c & d \\ a^2 & b^2 & c^2 & d^2 \\ a^4 & b^4 & c^4 & d^4 \end{vmatrix}$; (5) $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix}$; (6) $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^4 & b^4 & c^4 \end{vmatrix}$.

ART. 17. CO-FACTORS; MINORS.

The terms of $\Delta \equiv |a_1^{(n)}|$ which contain the element a_1' may be obtained by expanding the determinant

$$\begin{vmatrix} a_1' & 0 & 0 & \dots & 0 \\ a_2' & a_2'' & a_2''' & \dots & a_2^{(n)} \\ \dots & \dots & \dots & \dots & \dots \\ a_n' & a_n'' & a_n''' & \dots & a_n^{(n)} \end{vmatrix}. \qquad (1)$$

For, in writing out this expansion each term is formed by taking one, and only one, element from each column and each

* These determinants belong to an important class known as alternants. See Hanus' Elements of Determinants, Boston, 1888, pp. 187–201.

row of the array (Art. 7). If, therefore, in selecting the elements for any term, any other element than a_1' be taken from the first column, the one taken from the first row must be zero. Hence, the only terms which do not vanish are those which contain the element a_1'.

Moreover, in the terms of the expansion of (1) which do not vanish, a_1' is multiplied by $(n-1)$ elements chosen one from each column and each row of

$$\begin{vmatrix} a_2'' & a_2''' \ldots a_2^{(n)} \\ \ldots & \ldots \\ a_n'' & a_n''' \ldots a_n^{(n)} \end{vmatrix}. \tag{2}$$

There are $(n-1)!$ such terms, any one of which may be written $\pm a_1'a_i''a_j''' \ldots a_l^{(n)}$; the sign being determined by the class of the permutation of the n subscripts $1, i, j, \ldots l$. But since this is of the same class as the permutation of the $(n-1)$ subscripts $i, j, \ldots l$, the sign of any term, $\pm a_1'a_i''a_j''' \ldots a_l^{(n)}$, of the expansion of (1) is the same as the sign of the corresponding term, $a_i''a_j''' \ldots a_l^{(n)}$, of the expansion of (2). Hence,

$$\begin{vmatrix} a_1' & 0 & 0 & \ldots 0 \\ a_2' & a_2'' & a_2''' & \ldots a_2^{(n)} \\ \ldots & \ldots & \ldots & \ldots \\ a_n' & a_n'' & a_n''' & \ldots a_n^{(n)} \end{vmatrix} = a_1' \begin{vmatrix} a_2'' & a_2''' \ldots a_2^{(n)} \\ \ldots & \ldots \\ a_n'' & a_n''' \ldots a_n^{(n)} \end{vmatrix}. \tag{3}$$

The determinant (2) is called the co-factor or complement of the element a_1' in the determinant $|a_1^{(n)}|$. It is obtained from this determinant by deleting the first column and the first row.

The co-factor of any element $a_k^{(s)}$ may be found in the same manner upon transposing this element to the leading position. But by this transposition the sign of the determinant will be changed or not according as $a_k^{(s)}$ occupies a negative or a positive position (Art. 11). Hence, to find the co-factor of any element $a_s^{(k)}$ of the determinant $|a_1^{(n)}|$, delete the row and the column to which the element belongs, giving the resulting determinant the $\left\{\begin{matrix}\text{positive}\\\text{negative}\end{matrix}\right\}$ sign when $(k+s)$ is $\left\{\begin{matrix}\text{even}\\\text{odd}\end{matrix}\right\}$.

The co-factor thus obtained is represented by the symbol

$$A_k^{(s)};$$

the sign-factor of which, $(-1)^{k+s}$, is intrinsic, i.e., included in the symbol itself, which is accordingly written as positive. The co-factors of the various elements of $|a_{11}a_{22}a_{33}|$ are as follows:

$$A_{11} \equiv \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}; \quad A_{12} \equiv - \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}; \quad A_{13} \equiv \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix};$$

$$A_{21} \equiv - \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix}; \quad A_{22} \equiv \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix}; \quad A_{23} \equiv - \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix};$$

$$A_{31} \equiv \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}; \quad A_{32} \equiv - \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}; \quad A_{33} \equiv \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}.$$

The result obtained by deleting the kth row and the sth column of $\Delta \equiv |a_1^{(n)}|$ is called the minor of the determinant with respect to the element $a_k^{(s)}$, and is written $\Delta\{^s_k$. This minor is the same as the co-factor of the same element without its sign-factor; thus:

$$A_k^{(s)} = (-1)^{k+s}\Delta\{^s_k.$$

Similarly $\Delta\{^{p,s}_{h,k}$ is the result obtained by deleting the hth and kth rows and the pth and sth columns of Δ, and is called a second minor of the given determinant. Minors of still lower orders are obtained in a similar manner, and expressed by a similar notation. The kth minors are determinants of the order $(n-k)$.

Art. 18. Development in Terms of Co-factors.

The $(n-1)!$ terms of $|a_1^{(n)}|$ which contain $a_k^{(s)}$ are represented in the aggregate by $a_k^{(s)}A_k^{(s)}$ (Eq. 3, Art. 17). In like manner the groups of terms containing the successive elements $a_k', a_k'', \ldots a_k^{(n)}$ are respectively

$$a_k'A_k', \quad a_k''A_k'', \ldots a_k^{(n)}A_k^{(n)}.$$

Each one of these n groups includes $(n-1)!$ terms of the determinant $|a_1^{(n)}|$, no one of which is found in any other

group. In all of them, then, there are $n \times (n-1)!$ or $n!$ different terms of the determinant, which is the whole number. Hence,

$$| a_1^{(n)} | = a_k' A_k' + a_k'' A_k'' + \ldots + a_k^{(n)} A_k^{(n)}. \qquad (1)$$

Similarly (Art. 10),

$$| a_1^{(n)} | = a_1^{(s)} A_1^{(s)} + a_2^{(s)} A_2^{(s)} + \ldots + a_n^{(s)} A_n^{(s)}. \qquad (2)$$

Any determinant may, by means of either (1) or (2), be resolved into determinants of an order one lower. Since, in these formulas $A_k', \ldots A_k^{(n)}$, or $A_1^{(s)}, \ldots A_n^{(s)}$ are themselves determinants, they may be resolved into determinants of an order still one lower in the same manner. By continuing the process any determinant may ultimately be expressed in terms of determinants of the third or second order, which may be easily expanded by methods already given (Art. 9).

For example, let it be required to develop the determinant $\Delta \equiv | a_1 \, b_2 \, c_3 \, d_4 |$. Applying formula (1), letting $k = 1$, gives

$$\Delta = a_1 \begin{vmatrix} b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \\ b_4 & c_4 & d_4 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 & d_2 \\ a_3 & c_3 & d_3 \\ a_4 & c_4 & d_4 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \\ a_4 & b_4 & d_4 \end{vmatrix} - d_1 \begin{vmatrix} a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \\ a_4 & b_4 & c_4 \end{vmatrix}.$$

Upon a second application of the same formula this becomes

$$\begin{aligned} \Delta = {} & a_1 b_2 \begin{vmatrix} c_3 & d_3 \\ c_4 & d_4 \end{vmatrix} - a_1 c_2 \begin{vmatrix} b_3 & d_3 \\ b_4 & d_4 \end{vmatrix} + a_1 d_2 \begin{vmatrix} b_3 & c_3 \\ b_4 & c_4 \end{vmatrix} \\ & - a_2 b_1 \begin{vmatrix} c_3 & d_3 \\ c_4 & d_4 \end{vmatrix} + b_1 c_2 \begin{vmatrix} a_3 & d_3 \\ a_4 & d_4 \end{vmatrix} - b_1 d_2 \begin{vmatrix} a_3 & c_3 \\ a_4 & c_4 \end{vmatrix} \\ & + a_2 c_1 \begin{vmatrix} b_3 & d_3 \\ b_4 & d_4 \end{vmatrix} - b_2 c_1 \begin{vmatrix} a_3 & d_3 \\ a_4 & d_4 \end{vmatrix} + c_1 d_2 \begin{vmatrix} a_3 & b_3 \\ a_4 & b_4 \end{vmatrix} \\ & - a_2 d_1 \begin{vmatrix} b_3 & c_3 \\ b_4 & c_4 \end{vmatrix} + b_2 d_1 \begin{vmatrix} a_3 & c_3 \\ a_4 & c_4 \end{vmatrix} - c_2 d_1 \begin{vmatrix} a_3 & b_3 \\ a_4 & b_4 \end{vmatrix}. \end{aligned}$$

The complete development may be written out directly from the above. It is given in Eq. 3, Art. 7.

Prob. 19. Develop the following determinants:

$$(1)\begin{vmatrix} 1 & x & 1 & y \\ x & 1 & y & 1 \\ 1 & y & 1 & x \\ y & 1 & x & 1 \end{vmatrix}; \quad (2)\begin{vmatrix} a & x & y & a \\ x & 0 & 0 & y \\ y & 0 & 0 & x \\ a & y & x & a \end{vmatrix}; \quad (3)\begin{vmatrix} 0 & q & r & s \\ p & 0 & r & s \\ p & q & 0 & s \\ p & q & r & 0 \end{vmatrix}.$$

(Ans. $(x-y)^2((x+y)^2-4)$; $(x^2-y^2)^2$; $-3pqrs$.)

Prob. 20. Find the values of the following determinants:

$$(1)\begin{vmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \\ 3 & 4 & 1 & 2 \\ 4 & 1 & 2 & 3 \end{vmatrix}; \quad (2)\begin{vmatrix} 0 & 1 & 0 & 2 \\ 1 & 0 & 2 & 0 \\ 0 & 2 & 0 & 1 \\ 2 & 0 & 1 & 0 \end{vmatrix}; \quad (3)\begin{vmatrix} 3 & 5 & 3 & 1 \\ 6 & 6 & -1 & 1 \\ 9 & -3 & 5 & 1 \\ 8 & 3 & 0 & 1 \end{vmatrix};$$

$$(4)\begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{vmatrix}; \quad (5)\begin{vmatrix} 3 & 3 & 3 & 3 \\ 3 & 2 & 2 & 2 \\ 2 & 2 & 1 & 1 \\ 1 & 1 & 1 & 0 \end{vmatrix}; \quad (6)\begin{vmatrix} 0 & 0 & 0 & 3 \\ 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{vmatrix}.$$

(Ans. 160; 9; 0; -3; -3; -3.)

Prob. 21. Obtain the determinants in Exs. 5 and 6 of the preceding problem from that in Ex. 4.

Prob. 22. Evaluate $\begin{vmatrix} 0 & 1 & 1 & \ldots \\ 1 & 0 & 1 & \ldots \\ 1 & 1 & 0 & \ldots \\ \ldots & \ldots & \ldots & \ldots \end{vmatrix}$, of the nth order.

(Ans. $(n-1)(-1)^{n-1}$.)

Prob. 23. Show that $\begin{vmatrix} a & b & c & d \\ -b & a & -d & c \\ -c & d & a & -b \\ -d & -c & b & a \end{vmatrix} = (a^2+b^2+c^2+d^2)^2.$

ART. 19. THE ZERO FORMULAS.

If in the determinant $|a_1^{(n)}|$ the hth and kth rows be supposed identical, the elements $a_k', a_k'', \ldots a_k^{(n)}$ in the formula (1) of the last article may be replaced by $a_h', a_h'', \ldots a_h^{(n)}$ respectively. But in this case the value of the determinant is zero (Art. 12). Hence, in reference to the determinant $|a_1^{(n)}|$, h and k being different subscripts,

$$a_h'A_k' + a_h''A_k'' + \ldots + a_h^{(n)}A_k^{(n)} = 0.$$

Similarly, p and s being different superscripts,

$$a_1^{(p)}A_1^{(s)} + a_2^{(p)}A_2^{(s)} + \ldots + a_n^{(p)}A_n^{(s)} = 0.$$

Art. 20. Cauchy's Method of Development.

It is frequently desirable to expand a determinant with reference to the elements of a given row and column.

Let the determinant be $\Delta \equiv | a_1^{(n)} |$, and the given row and column the hth and pth respectively. Then is $A_h^{(p)}$ the co-factor of $a_h^{(p)}$, the element at the intersection of the two given lines. The co-factor of any element $a_k^{(s)}$ of $A_h^{(p)}$ will be designated by $B_k^{(s)}$, this being a determinant of the order $(n-2)$. The required expansion may now be obtained by means of the following formula, due to Cauchy:

$$| a_1^{(n)} | = a_h^{(p)}A_h^{(p)} - \Sigma a_h^{(s)}a_k^{(p)}B_k^{(s)}, \qquad (1)$$

in which $k = 1, 2, \ldots h-1, h+1, \ldots n$, and $s = 1, 2, \ldots p-1, p+1, \ldots n$, successively.

To prove this, consider that $B_k^{(s)}$ is the aggregate of all terms of the expansion of Δ which contain the product $a_h^{(p)}a_k^{(s)}$. These terms are included in $a_h^{(p)}A_h^{(p)}$. Now, every term in the expansion which does not contain $a_h^{(p)}$ must contain some other element $a_h^{(s)}$ from the hth row and also some other element $a_k^{(p)}$ from the pth column, and thus contains the product $a_h^{(s)}a_k^{(p)}$. But this product differs from $a_h^{(p)}a_k^{(s)}$ only in the order of the superscripts; and is, therefore, in the expansion of Δ, multiplied by an aggregate of terms differing in sign only from that multiplying $a_h^{(p)}a_k^{(s)}$. Hence, $-a_h^{(p)}a_k^{(s)}B_k^{(s)}$ is the coefficient of $a_h^{(s)}a_k^{(p)}$ in the required expansion.

In the formula $a_h^{(p)}A_h^{(p)}$ gives $(n-1)!$ terms of Δ. There are also $(n-1)^2$ such aggregates as $-a_h^{(s)}a_k^{(p)}B_k^{(s)}$, each containing $(n-2)!$ terms. The formula therefore gives $(n-1)! + (n-1)^2(n-2)! = n!$ terms, which is the complete expansion.

When the expansion is required with reference to the ele-

ments of the first column and the first row the formula, written explicitly, becomes

$$
\begin{aligned}
|a_1^{(n)}| = a_1'A_1' - a_2'a_1''B_2'' - a_2'a_1'''B_2''' - \ldots - a_2'a_1^{(n)}B_2^{(n)} \\
- a_3'a_1''B_3'' - a_3'a_1'''B_3''' - \ldots - a_3'a_1^{(n)}B_3^{(n)} \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
- a_n'a_1''B_n'' - a_n'a_1'''B_n''' - \ldots - a_n'a_1^{(n)}B_n^{(n)}, \quad (2)
\end{aligned}
$$

in which $B_k^{(s)}$ has intrinsically the sign $(-1)^{k+s}$.

Cauchy's formula is particularly useful in expanding determinants which have been bordered; such as

$$
-Q = \begin{vmatrix} 0 & u_1 & u_2 & u_3 \\ u_1 & a_{11} & a_{12} & a_{13} \\ u_2 & a_{21} & a_{22} & a_{23} \\ u_3 & a_{31} & a_{32} & a_{33} \end{vmatrix}. \quad (3)
$$

Applying formula (2) to this determinant gives

$$
\begin{aligned}
-Q = -u_1^2 \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + u_1u_2 \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} - u_3u_1 \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \\
+ u_1u_2 \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} - u_2^2 \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} + u_2u_3 \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} \\
- u_3u_1 \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} + u_2u_3 \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} - u_3^2 \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}.
\end{aligned}
$$

Letting $a_{ks} \equiv a_{sk}$, and writing $A_{11}, A_{12}, \ldots$ for the co-factors of the elements of $|a_{11}a_{22}a_{33}|$, the above becomes

$$
Q = A_{11}u_1^2 + A_{22}u_2^2 + A_{33}u_3^2 + 2A_{23}u_2u_3 + 2A_{31}u_3u_1 + 2A_{12}u_1u_2.
$$

Prob. 24. Develop the following determinants by Cauchy's formula:

$$
(1)\ -\begin{vmatrix} a & h & g & u \\ h & b & f & v \\ g & f & c & w \\ u & v & w & 0 \end{vmatrix}; \qquad
(2)\ \begin{vmatrix} 0 & yz & zx & xy \\ yz & 0 & 1 & 1 \\ zx & 1 & 0 & 1 \\ xy & 1 & 1 & 0 \end{vmatrix}; \qquad
(3)\ \begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & xy & zx \\ 1 & xy & 0 & yz \\ 1 & zx & yz & 0 \end{vmatrix};
$$

(4) $\begin{vmatrix} -1 & -x & 1 & 1 \\ 1 & -y & -1 & 1 \\ x & 0 & y & z \\ 1 & -z & 1 & -1 \end{vmatrix}$; (5) $\begin{vmatrix} 1 & 1 & 1 & x \\ x & y & z & 0 \\ 1 & 1 & 1 & y \\ 1 & 1 & 1 & z \end{vmatrix}$; (6) $\begin{vmatrix} 0 & a & b \\ -a & \sin A & \sin B \\ -b & -\cos A & \cos B \end{vmatrix}$.

ART. 21. DIFFERENTIATION OF DETERMINANTS.

By the formula (1) of Art. 18

$$\Delta \equiv | y_{1,n} | = Y_{k1} y_{k1} + Y_{k2} y_{k2} + \dots Y_{kn} y_{kn}. \qquad (1)$$

Considering the elements of the determinant as independent variables and differentiating with respect to y_{ks} gives

$$\delta_{ks}\Delta = Y_{ks}\, dy_{ks}, \quad \text{or} \quad Y_{ks} = \frac{\delta \Delta}{dy_{ks}}. \qquad (2)$$

Substituting in (1),

$$\Delta \equiv | y_{1,n} | = y_{k1}\frac{\delta \Delta}{dy_{k1}} + y_{k2}\frac{\delta \Delta}{dy_{k2}} + \dots + y_{kn}\frac{\delta \Delta}{dy_{kn}}. \qquad (3)$$

Similarly

$$\Delta \equiv | y_{1,n} | = y_{1s}\frac{\delta \Delta}{dy_{1s}} + y_{2s}\frac{\delta \Delta}{dy_{2s}} + \dots + y_{ns}\frac{\delta \Delta}{dy_{ns}}. \qquad (4)$$

Again differentiating (1), this time with respect to all the elements of the kth row, there results

$$\delta_k \Delta = Y_{k1} dy_{k1} + Y_{k2} dy_{k2} + \dots + Y_{kn} dy_{kn}. \qquad (5)$$

In the total differential of Δ there are obviously n such expressions as (5), each of which may be obtained from Δ by replacing the elements of some one of the rows by their differentials; thus:

$$d\Delta = \begin{vmatrix} dy_{11} \dots dy_{1n} \\ y_{21} \dots y_{2n} \\ \dots\dots\dots \\ y_{n1} \dots y_{nn} \end{vmatrix} + \begin{vmatrix} y_{11} \dots y_{1n} \\ dy_{21} \dots dy_{2n} \\ \dots\dots\dots \\ y_{n1} \dots y_{nn} \end{vmatrix} + \dots + \begin{vmatrix} y_{11} \dots y_{1n} \\ y_{21} \dots y_{2n} \\ \dots\dots\dots \\ dy_{n1} \dots dy_{nn} \end{vmatrix}. \qquad (6)$$

If all the elements are functions of one independent variable x, then, representing $\frac{dy_{ks}}{dx}$ by y_{ks}',

$$\frac{d\Delta}{dx} = \begin{vmatrix} y_{11}' \dots y_{1n}' \\ y_{21} \dots y_{2n} \\ \dots\dots\dots \\ y_{n1} \dots y_{nn} \end{vmatrix} + \begin{vmatrix} y_{11} \dots y_{1n} \\ y_{21}' \dots y_{2n}' \\ \dots\dots\dots \\ y_{n1} \dots y_{nn} \end{vmatrix} + \dots + \begin{vmatrix} y_{11} \dots y_{1n} \\ y_{21} \dots y_{2n} \\ \dots\dots\dots \\ y_{n1}' \dots y_{nn}' \end{vmatrix}. \qquad (7)$$

Prob. 25. Show that Cauchy's formula may be written

$$\Delta \equiv |a_1^{(n)}| = a_h^{(p)} \frac{\delta \Delta}{da_h^{(p)}} - \Sigma a_k^{(p)} a_h^{(s)} \frac{\delta^2 \Delta}{da_h^{(p)} da_k^{(s)}}.$$

ART. 22. RAISING THE ORDER.

Since, in the expansion of the determinant (1) of Art. 17 the elements $a_2', \ldots a_n'$ do not appear, these may be replaced by any quantities whatever, as $Q, \ldots T$, without changing the value of the determinant; thus:

$$\begin{vmatrix} a_1' & 0 & 0 & \ldots & 0 \\ a_2' & a_2'' & a_2''' & \ldots & a_2^{(n)} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ a_n' & a_n'' & a_n''' & \ldots & a_n^{(n)} \end{vmatrix} = \begin{vmatrix} a_1' & 0 & 0 & \ldots & 0 \\ Q & a_2'' & a_2''' & \ldots & a_2^{(n)} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ T & a_n'' & a_n''' & \ldots & a_n^{(n)} \end{vmatrix}.$$

Similarly,

$$\begin{vmatrix} a_1' & 0 & 0 & \ldots & 0 \\ a_2' & a_2'' & 0 & \ldots & 0 \\ a_3' & a_3'' & a_3''' & \ldots & a_3^{(n)} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ a_n' & a_n'' & a_n''' & \ldots & a_n^{(n)} \end{vmatrix} = a_1'a_2'' \begin{vmatrix} a_3''' & \ldots & a_3^{(n)} \\ \ldots & \ldots & \ldots \\ a_n''' & \ldots & a_n^{(n)} \end{vmatrix} = \begin{vmatrix} a_1' & 0 & 0 & \ldots & 0 \\ Q & a_2'' & 0 & \ldots & 0 \\ R & L & a_3''' & \ldots & a_3^{(n)} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ T & N & a_n''' & \ldots & a_n^{(n)} \end{vmatrix},$$

in which $Q, R, \ldots T$ and $L, \ldots N$ are any quantities whatever. Finally,

$$\begin{vmatrix} a_1' & 0 & \ldots 0 & 0 \\ a_2' & a_2'' & \ldots 0 & 0 \\ \ldots & \ldots & \ldots & \ldots \\ a'_{n-1} & a''_{n-1} & \ldots a_{n-1}^{(n-1)} & 0 \\ a_n & a_n'' & \ldots a_n^{(n-1)} & a_n^{(n)} \end{vmatrix} = a_1'a_2'' \ldots a_{n-1}^{(n-1)} a_n^{(n)} = \begin{vmatrix} a_1' & 0 & \ldots 0 & 0 \\ Q & a_2'' & \ldots 0 & 0 \\ \ldots & \ldots & \ldots & \ldots \\ S & M & \ldots a_{n-1}^{(n-1)} & 0 \\ T & N & \ldots C & a_n^{(n)} \end{vmatrix};$$

that is, if all the elements on one side of the principal diagonal are zeros the determinant is equal to its principal term, and the elements on the other side of this diagonal may be replaced by any quantities whatever.

By what precedes,

$$\begin{vmatrix} a_1' & \ldots & a_1^{(n)} \\ \ldots & \ldots & \ldots \\ a_n' & \ldots & a_n^{(n)} \end{vmatrix} = \begin{vmatrix} 1 & 0 & \ldots & 0 \\ Q & a_1' & \ldots & a_1^{(n)} \\ \ldots & \ldots & \ldots & \ldots \\ T & a_n' & \ldots & a_n^{(n)} \end{vmatrix}$$

Hence, a determinant of the nth order may be expressed as a determinant of the order $(n+1)$ by bordering it above by a row (to the left by a column) of zeros, to the left by a column (above by a row) of elements chosen arbitrarily, and writing 1 at the intersection of the lines thus added. By continuing this process any determinant may be expressed as a determinant of any higher order.

Prob. 26. If all the elements on one side of the secondary diagonal are zeros, what is the value of the determinant?

Prob. 27. Develop the determinant $\begin{vmatrix} a & h & g & u & 0 \\ h & b & f & v & 0 \\ g & f & c & w & 0 \\ u & v & w & 0 & t \\ 0 & 0 & 0 & t & s \end{vmatrix}$.

Prob. 28. A determinant in which $a_k^{(s)} = -a_s^{(k)}$ and $a_k^{(k)} = 0$ is said to be skew-symmetric. Prove that every skew-symmetric determinant of odd order is equal to zero.

Art 23. Lowering the Order.

The following method of reducing and evaluating a determinant is often useful, particularly when the elements are numerical. Let the determinant be

$$\Delta \equiv \begin{vmatrix} a_1' a_1'' \dots a_1^{(n)} \\ a_2' a_2'' \dots a_2^{(n)} \\ \dots\dots\dots\dots \\ a_n' a_n'' \dots a_n^{(n)} \end{vmatrix}. \quad \text{Then}$$

$$\Delta = \frac{1}{(a_1')^{n-1}} \begin{vmatrix} a_1' & a_1'a_1'' & a_1'a_1''' \dots \\ a_2' & a_1'a_2'' & a_1'a_2''' \dots \\ a_3' & a_1'a_3'' & a_1'a_3''' \dots \\ \dots & \dots & \dots \end{vmatrix} \qquad \text{(Art. 13.)}$$

$$= \frac{1}{(a_1')^{n-2}} \begin{vmatrix} a_1'a_2'' - a_2'a_1'' & a_1'a_2''' - a_2'a_1''' \dots \\ a_1'a_3'' - a_3'a_1'' & a_1'a_3''' - a_3'a_1''' \dots \\ \dots & \dots \end{vmatrix}. \qquad \text{(Arts. 15, 17.)}$$

In this last expression the determinant is obviously of the order $(n-1)$. The process may be formulated thus: Replace each element $a_k^{(s)}$ of the minor of the leading element by the

determinant $\begin{vmatrix} a_1' & a_1^{(s)} \\ a_k' & a_k^{(s)} \end{vmatrix}$, and the resulting determinant divided by $(a_1')^{n-2}$ will be equal to the given determinant. The elements being numerical the process may be repeated with ease until the order becomes unity. The example given below will illustrate.

$$\Delta \equiv \begin{vmatrix} 1 & 2 & 3 & 4 \\ 8 & 7 & 6 & 5 \\ 1 & 8 & 2 & 7 \\ 3 & 6 & 4 & 5 \end{vmatrix} = \begin{vmatrix} \begin{vmatrix} 1 & 2 \\ 8 & 7 \end{vmatrix} & \begin{vmatrix} 1 & 3 \\ 8 & 6 \end{vmatrix} & \begin{vmatrix} 1 & 4 \\ 8 & 5 \end{vmatrix} \\ \begin{vmatrix} 1 & 2 \\ 1 & 8 \end{vmatrix} & \begin{vmatrix} 1 & 3 \\ 1 & 2 \end{vmatrix} & \begin{vmatrix} 1 & 4 \\ 1 & 7 \end{vmatrix} \\ \begin{vmatrix} 1 & 2 \\ 3 & 6 \end{vmatrix} & \begin{vmatrix} 1 & 3 \\ 3 & 4 \end{vmatrix} & \begin{vmatrix} 1 & 4 \\ 3 & 5 \end{vmatrix} \end{vmatrix}$$

$$= \begin{vmatrix} -9 & -18 & -27 \\ 6 & -1 & 3 \\ 0 & -5 & -7 \end{vmatrix} = \begin{vmatrix} \begin{vmatrix} -9 & -18 \\ 6 & -1 \end{vmatrix} & \begin{vmatrix} -9 & -27 \\ 6 & 3 \end{vmatrix} \\ \begin{vmatrix} -9 & -18 \\ 0 & -5 \end{vmatrix} & \begin{vmatrix} -9 & -27 \\ 0 & -7 \end{vmatrix} \end{vmatrix}$$

$$= \begin{vmatrix} 117 & 135 \\ 45 & 63 \end{vmatrix} = 1296.$$

ART. 24. SOLUTION OF LINEAR EQUATIONS.

Of the many analytical processes giving rise to determinants the simplest and most common is the solution of systems of simultaneous linear equations. Thus, solving the equations

$$\left. \begin{aligned} a_1'x' + a_1''x'' &= \kappa_1, \\ a_2'x' + a_2''x'' &= \kappa_2, \end{aligned} \right\}$$

by the methods of ordinary algebra gives:

$$x' = \frac{\kappa_1 a_2'' - \kappa_2 a_1''}{a_1'a_2'' - a_2'a_1''}, \quad x'' = \frac{a_1'\kappa_2 - a_2'\kappa_1}{a_1'a_2'' - a_2'a_1''}.$$

In the notation of determinants these are written:

$$x' = \begin{vmatrix} \kappa_1 & a_1'' \\ \kappa_2 & a_2'' \end{vmatrix} \Big/ \begin{vmatrix} a_1' & a_1'' \\ a_2' & a_2'' \end{vmatrix}, \quad x'' = \begin{vmatrix} a_1' & \kappa_1 \\ a_2' & \kappa_2 \end{vmatrix} \Big/ \begin{vmatrix} a_1' & a_1'' \\ a_2' & a_2'' \end{vmatrix}.$$

It will be noted that the two fractions expressing the values of x' and x'' have a common denominator, this being the determinant whose elements are the coefficients of the unknowns arranged in the same order as in the given equations. The

numerator of the fraction giving the value of x' is formed from this denominator by replacing each coefficient of x' by the corresponding absolute term. Similarly for x''.

The difficulty of solving systems of linear equations by the ordinary processes of elimination increases rapidly as the number of equations is increased. The law of formation of the roots explained above is. however, capable of generalization, being equally applicable to all complete linear systems, as will now be shown.

Let such a system be written

$$\left.\begin{array}{l} a_1'x' + a_1''x'' + \ldots + a_1^{(n)}x^{(n)} = \kappa_1, \\ a_2'x' + a_2''x'' + \ldots + a_2^{(n)}x^{(n)} = \kappa_2, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_n'x' + a_n''x'' + \ldots + a_n^{(n)}x^{(n)} = \kappa_n. \end{array}\right\}. \qquad (1)$$

Now form the determinant of the coefficients of these equations; thus:

$$D \equiv \begin{vmatrix} a_1'a_1'' \ldots a_1^{(n)} \\ a_2'a_2'' \ldots a_2^{(n)} \\ \ldots\ldots\ldots\ldots \\ a_n'a_n'' \ldots a_n^{(n)} \end{vmatrix},$$

and let $A_k^{(s)}$ be the co-factor of $a_k^{(s)}$ in this determinant. The function

$$a_1^{(p)}A_1^{(s)} + a_2^{(p)}A_2^{(s)} + \ldots + a_n^{(p)}A_n^{(s)}$$

is equal to D when $p = s$ (Art. 18); to zero when p and s are different superscripts (Art. 19). Then, multiplying the given equations by $A_1^{(s)}$, $A_2^{(s)}$, ... $A_n^{(s)}$ respectively, the sum of the resulting equations is a linear equation in which the coefficient of $x^{(s)}$ is equal to D, while those of all the other unknowns vanish. The sum is, therefore,

$$Dx^{(s)} = \kappa_1 A_1^{(s)} + \kappa_2 A_2^{(s)} + \ldots + \kappa_n A_n^{(s)}. \qquad (2)$$

But the second member of this equation is what D becomes upon replacing the coefficients $a_1^{(s)}$, $a_2^{(s)}$, ... $a_n^{(s)}$ of the unknown $x^{(s)}$ by the absolute terms κ_1, κ_2, ... κ_n in order. Hence,

$$x^{(s)} = \begin{vmatrix} a_1' \ldots a_1^{(s-1)}\kappa_1 a_1^{(s+1)} \ldots a_1^{(n)} \\ a_2' \ldots a_2^{(s-1)}\kappa_2 a_2^{(s+1)} \ldots a_2^{(n)} \\ \ldots\ldots\ldots\ldots\ldots \\ a_n' \ldots a_n^{(s-1)}\kappa_n a_n^{(s+1)} \ldots a_n^{(n)} \end{vmatrix} \Bigg/ \begin{vmatrix} a_1'a_1'' \ldots a_1^{(n)} \\ a_2'a_2'' \ldots a_2^{(n)} \\ \ldots\ldots\ldots \\ a_n'a_n'' \ldots a_n^{(n)} \end{vmatrix}. \qquad (3)$$

This result may be stated as follows:

(*a*) The common denominator of the fractions expressing the values of the unknowns in a system of n linear equations involving n unknown quantities is the determinant of the coefficients, these being written in the same order as in the given equations. (*b*) The numerator of the fraction giving the value of any one of the unknowns is a determinant, which may be formed from the determinant of the coefficients by substituting for the column made up of the coefficients of the unknown in question a column whose elements are the absolute terms of the equations taken in the same order as the coefficients which they displace.

Prob. 29. Solve the following systems of equations:

(1) $3x + 5y = 21, \quad 6x + 2y = 15;$

(2) $\dfrac{x}{3} + \dfrac{3y}{2} = 5, \quad \dfrac{2x}{3} + y = 6;$

(3) $3x + y + 2z = 50, \quad x + 2y - 3z = 15, \quad 2x + 2y - 3z = 25;$

(4) $\dfrac{1}{y} + \dfrac{1}{z} = p, \quad \dfrac{1}{z} + \dfrac{1}{x} = q, \quad \dfrac{1}{x} + \dfrac{1}{y} = r;$

(5) $\dfrac{w}{3} + \dfrac{x}{5} + \dfrac{y}{7} + \dfrac{z}{9} = 2800, \quad \dfrac{w}{5} + \dfrac{x}{7} + \dfrac{y}{9} + \dfrac{z}{11} = 2144,$

$\dfrac{w}{7} + \dfrac{x}{9} + \dfrac{y}{11} + \dfrac{z}{13} = 1744, \quad \dfrac{w}{9} + \dfrac{x}{11} + \dfrac{y}{13} + \dfrac{z}{15} = 1472.$

Prob. 30. Show that the three right lines

$$y = x + 1, \quad y = -2x + 16, \quad y = 3x - 9,$$

intersect in a common point.

Art. 25. Consistence of Linear Systems.

When the number of given equations is greater than the number of unknowns their consistency with one another must

obviously depend upon some relation among the coefficients. This relation will now be investigated for the case of $(n+1)$ linear equations involving n unknowns. Let the equations be

$$\left.\begin{array}{l} a_1'x' + \ldots + a_1^{(n)}x^{(n)} = \kappa_1, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_n'x' + \ldots + a_n^{(n)}x^{(n)} = \kappa_n, \\ a_{n+1}'x' + \ldots + a_{n+1}^{(n)}x^{(n)} = \kappa_{n+1}. \end{array}\right\}$$

If the above equations are consistent the values of the unknowns obtained by solving any n of them must satisfy the remaining equation. Solving the first n equations by the method of the preceding article, substituting the values of $x', x'', \ldots x^{(n)}$ thus obtained in the last equation, and clearing of fractions, the result reduces to (Art. 18)

$$E \equiv \begin{vmatrix} a_1' & \ldots a_1^{(n)} & \kappa_1 \\ \ldots & \ldots & \ldots \\ a_n' & \ldots a_n^{(n)} & \kappa_n \\ a_{n+1}' & \ldots a_{n+1}^{(n)} & \kappa_{n+1} \end{vmatrix} = 0,$$

which is the condition to be fulfilled by the coefficients in order that the given equations may be consistent.

Hence the condition of consistency for a set of linear equations involving a number of unknowns one less than the number of equations is that the determinant of the coefficients and absolute terms, written in the same order as in the given equations, shall be zero. This determinant is called the resultant* or eliminant of the equations. Thus the equations

$$x+y-z=0,\ x-y+z=2,\ -x+y+z=4,\ x+y+z=6$$

are consistent, for the reason that

$$\begin{vmatrix} 1 & 1 & -1 & 0 \\ 1 & -1 & 1 & 2 \\ -1 & 1 & 1 & 4 \\ 1 & 1 & 1 & 6 \end{vmatrix} = 0.$$

* This term was introduced by Laplace in 1772. The term eliminant is due to Sylvester.

Art. 26. The Matrix.

Assume r linear equations involving n unknowns, r being greater than n as follows:

$$\left.\begin{array}{l} a_1'x' + \ldots + a_1^{(n)}x^{(n)} = \kappa_1, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_n'x' + \ldots + a_n^{(n)}x^{(n)} = \kappa_n, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_r'x' + \ldots + a_r^{(n)}x^{(n)} = \kappa_r. \end{array}\right\}$$

The consistency of these equations requires that every determinant of the order $(n + 1)$, formed by selecting $(n + 1)$ rows from the array whose elements are the coefficients and absolute terms written in order, shall be zero.

If the elements of the array fulfill this condition the fact is expressed thus:

$$\left\|\begin{array}{lll} a_1' & \ldots a_n' & \ldots a_r' \\ \ldots & \ldots & \ldots \\ a_1^{(n)} & \ldots a_n^{(n)} & \ldots a_r^{(n)} \\ \kappa_1 & \ldots \kappa_n & \ldots \kappa_r \end{array}\right\| = \cancel{0};$$ is of rank $\leq n$

the change of rows into columns being purely arbitrary. The above expression is called a rectangular array, or a matrix.

Art. 27. Homogeneous Linear Systems.

Let the equations of the given system be both linear and homogeneous; thus:

$$\left.\begin{array}{l} a_1'x' + \ldots + a_1^{(n)}x^{(n)} = 0, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_n'x' + \ldots + a_n^{(n)}x^{(n)} = 0. \end{array}\right\} \quad (1)$$

Representing the determinant of the coefficients by E, the general solution, as given by the formula (3) of Art. 24, is

$$x^{(s)} = 0/E.$$

That is, all the unknowns are equal to zero, and the equations have no other solution than this unless

$$E = 0.$$

But in this case the value of each unknown is obtainable only in the indeterminate form 0/0. The ratios of the unknowns may be readily obtained, however. For, dividing each equation through by any one of these, as $x^{(s)}$, the system (1) becomes

$$\left.\begin{array}{l} a_1'\dfrac{x'}{x^{(s)}}+\ldots+a_1^{(s-1)}\dfrac{x^{(s-1)}}{x^{(s)}}+a_1^{(s+1)}\dfrac{x^{(s+1)}}{x^{(s)}}+\ldots+a_1^{(n)}\dfrac{x^{(n)}}{x^{(s)}}=-a_1^{(s)}, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ a_n'\dfrac{x'}{x^{(s)}}+\ldots+a_n^{(s-1)}\dfrac{x^{(s-1)}}{x^{(s)}}+a_n^{(s+1)}\dfrac{x^{(s+1)}}{x^{(s)}}+\ldots+a_n^{(n)}\dfrac{x^{(n)}}{x^{(s)}}=-a_n^{(s)}. \end{array}\right\} \quad (2)$$

Now the condition $E=0$ establishes the consistency of the n equations (2) involving the $(n-1)$ unknown ratios (Art. 25),

$$\frac{x'}{x^{(s)}},\ \ldots\ \frac{x^{(s-1)}}{x^{(s)}},\ \frac{x^{(s+1)}}{x^{(s)}},\ \ldots\ \frac{x^{(n)}}{x^{(s)}}.$$

Hence, if $E=0$ the given equations (1) are consistent; that is, the values of the above $(n-1)$ ratios obtained by solving any $(n-1)$ of them will satisfy the remaining equation. Any n quantities having among themselves the ratios thus determined will satisfy the given equations. Thus, if $x_0', x_0'', \ldots x_0^{(n)}$ are n such quantities, so also are $\lambda x_0', \lambda x_0'', \ldots \lambda x_0^{(n)}$, λ being any factor whatever.

The determinant E of the coefficients of the given homogeneous linear equations is called the resultant or eliminant of the system.

When the number of equations is greater than the number of unknowns the conditions of consistency are expressible in the form of a rectangular array, as in Art. 26.

As an example, consider the five equations

$$2x-3y+z=0,\quad 4x-y-z=0,\quad -7x+3y+z=0,$$
$$x+y-z=0,\quad 5x-5y+z=0.$$

Dividing each of the first two equations by z and solving for the two unknowns $\frac{x}{z}$ and $\frac{y}{z}$ gives

$$\frac{x}{z}=\begin{vmatrix}-1 & -3\\ 1 & -1\end{vmatrix}\Big/\begin{vmatrix}2 & -3\\ 4 & -1\end{vmatrix}=\frac{2}{5},\quad \frac{y}{z}=\begin{vmatrix}2 & -1\\ 4 & 1\end{vmatrix}\Big/\begin{vmatrix}2 & -3\\ 4 & -1\end{vmatrix}=\frac{3}{5},$$

or $$x : y : z :: 2 : 3 : 5;$$
and any three quantities having these ratios will satisfy the two equations, as 10, 15, and 25. That the third equation is consistent with the first two is shown by the vanishing of the determinant

$$\begin{vmatrix} 2 & -3 & 1 \\ 4 & -1 & -1 \\ -7 & 3 & 1 \end{vmatrix} = 0.$$

If all the equations are consistent the determinant of the coefficients of any three of them must vanish; that is,

$$\begin{Vmatrix} 2 & 4 & -7 & 1 & 5 \\ -3 & -1 & 3 & 1 & -5 \\ 1 & -1 & 1 & -1 & 1 \end{Vmatrix} = 0.$$

ART. 28. CO-FACTORS IN A ZERO DETERMINANT.

If, in the preceding article, $E = 0$, it follows from Arts. 18 and 19 that

$$\begin{aligned} a_1'A_k' + a_1''A_k'' + \ldots + a_1^{(n)}A_k^{(n)} &= 0, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_k'A_k' + a_k''A_k'' + \ldots + a_k^{(n)}A_k^{(n)} &= 0 = E, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_n'A_k' + a_n''A_k'' + \ldots + a_n^{(n)}A_k^{(n)} &= 0. \end{aligned}$$

These equations obviously give for the ratios

$$\frac{A_k'}{A_k^{(s)}}, \ldots \frac{A_k^{(s-1)}}{A_k^{(s)}}, \quad \frac{A_k^{(s+1)}}{A_k^{(s)}}, \ldots \frac{A_k^{(n)}}{A_k^{(s)}}$$

values which are identical with those obtained for the ratios

$$\frac{x'}{x^{(s)}}, \ldots \frac{x^{(s-1)}}{x^{(s)}}, \quad \frac{x^{(s+1)}}{x^{(s)}}, \ldots \frac{x^{(n)}}{x^{(s)}}$$

from the equations (1) of Art. 27. It follows that $x', x'', \ldots x^{(n)}$ are proportional to $A_k', A_k'', \ldots A_k^{(n)}$, whatever the value of k. Thus, giving to k the successive values 1, 2, . . . n, there result

$$\begin{aligned} x' : x'' : \ldots : x^{(n)} &:: A_1' : A_1'' : \ldots : A_1^{(n)} \\ &:: A_2' : A_2'' : \ldots : A_2^{(n)} \\ &\ldots\ldots\ldots\ldots\ldots \\ &:: A_n' : A_n'' : \ldots : A_n^{(n)}. \end{aligned}$$

Hence, when a determinant is equal to zero, the co-factors of the elements of any line are proportional to the co-factors of the corresponding elements of any parallel line.

Art. 29. Sylvester's Method of Elimination.*

Let it be required to eliminate the unknown from the two equations

$$a_3x^3 + a_2x^2 + a_1x + a_0 = 0,$$
$$b_2x^2 + b_1x + b_0 = 0.$$

This will be done by what is called the dialytic method, the invention of which is due to Sylvester. Multiplying the first of the given equations by x, and the second by x and x^2 successively, the result is a system of five equations, viz.:

$$\left.\begin{array}{rr} a_3x^3 + a_2x^2 + a_1x + a_0 & = 0, \\ a_3x^4 + a_2x^3 + a_1x^2 + a_0x & = 0, \\ b_2x^2 + b_1x + b_0 & = 0, \\ b_2x^3 + b_1x^2 + b_0x & = 0, \\ b_2x^4 + b_1x^3 + b_0x^2 & = 0. \end{array}\right\}$$

The eliminant of these five equations, involving the four unknowns x, x^2, x^3, and x^4 is (Art. 25)

$$E \equiv \begin{vmatrix} 0 & a_3 & a_2 & a_1 & a_0 \\ a_3 & a_2 & a_1 & a_0 & 0 \\ 0 & 0 & b_2 & b_1 & b_0 \\ 0 & b_2 & b_1 & b_0 & 0 \\ b_2 & b_1 & b_0 & 0 & 0 \end{vmatrix} = 0.$$

If the given equations be not consistent this determinant will not vanish.

The above method is a general one. Thus, let the two given equations be

$$a_mx^m + \ldots + a_1x + a_0 = 0,$$
$$b_nx^n + \ldots\ldots + b_1x + b_0 = 0.$$

Multiplying the first equation $(n - 1)$ times in succession by x, and the second $(m - 1)$ times, $(m + n)$ equations are

* Philosophical Magazine, 1840, and Crelle's Journal, Vol. XXI.

obtained which involve as unknowns the first $(m+n-1)$ powers of x. The eliminant of these equations is a determinant of the order $(m+n)$, which is of the nth degree in terms of the coefficients of the equation of the mth degree, and *vice versa*. The law of formation of the eliminant is obvious.

The same method may be used in eliminating one or both the variables from a pair of homogeneous equations.

As an example, let it be required to eliminate the variables from the equations

$$2x^3 - 5x^2y - 9y^3 = 0 \quad \text{and} \quad 3x^2 - 7xy - 6y^2 = 0.$$

Dividing the first by y^3, and multiplying by $\frac{x}{y}$; the second by y^2, and multiplying by $\frac{x}{y}$ twice in succession, there result, in all, five equations involving $\frac{x}{y}$, $\frac{x^2}{y^2}$, $\frac{x^3}{y^3}$, and $\frac{x^4}{y^4}$. Eliminating these four ratios gives

$$E \equiv \begin{vmatrix} 0 & 2 & -5 & 0 & -9 \\ 2 & -5 & 0 & -9 & 0 \\ 0 & 0 & 3 & -7 & -6 \\ 0 & 3 & -7 & -6 & 0 \\ 3 & -7 & -6 & 0 & 0 \end{vmatrix},$$

the vanishing of which shows that the two given equations are consistent.

Prob. 31. Test the consistency of each of the following systems of equations:

(1) $x+y+2z=9$, $x+y-z=0$, $2x-y+z=3$, $x-3y+2z=1$;

(2) $x-y-2z=0$, $\quad x-2y+z=0$, $\quad 2x-3y-z=0$;

(3) $2x^2y-xy^2=0$, $\quad 8x^3y+8xy^3-5y^4=0$.

Prob. 32. Find the ratios of the unknowns in the equations

$$2x+y-2z=0, \quad 4w-y-4z=0, \quad 2w+x-5v+z=0.$$

Prob. 33. In the equations

$$a_k'x' + \ldots + a_k^{(n)}x^{(n)} + a_k^{(n+1)}x^{(n+1)} = 0, \quad [k=1, 2, \ldots n]$$

prove that $x' : \ldots : x^{(n)} : x^{(n+1)} :: M' : \ldots : M^{(n)} : M^{(n+1)}$, where

$(-1)^{i-1}M^{(i)}$ is the determinant obtained by deleting the ith column from the rectangular array

$$M \equiv \begin{vmatrix} a_1' \ldots a_1^{(n)} \, a_1^{(n+1)} \\ \cdots\cdots\cdots\cdots \\ a_n' \ldots a_n^{(n)} \, a_n^{(n+1)} \end{vmatrix}.$$

Prob. 34. From $\dfrac{lx + \nu y + \mu z}{p} = \dfrac{\nu x + my + \lambda z}{q} = \dfrac{\mu x + \lambda y + nz}{r}$,

deduce $\dfrac{x}{\begin{vmatrix} \nu & \mu & p \\ m & \lambda & q \\ \lambda & n & r \end{vmatrix}} = -\dfrac{y}{\begin{vmatrix} l & \mu & p \\ \nu & \lambda & q \\ \mu & n & r \end{vmatrix}} = \dfrac{z}{\begin{vmatrix} l & \nu & p \\ \nu & m & q \\ \mu & \lambda & r \end{vmatrix}}$.

Prob. 35. Show that the three straight lines $a'x + b'y + c' = 0$, $a''x + b''y + c'' = 0$, and $a'''x + b'''y + c''' = 0$, are concurrent when $| a'b''c''' | = 0$.

Prob. 36. Prove that the medians of a triangle are concurrent.

Prob. 37. Show that the points (x_0, y_0), (x_1, y_1), and (x_2, y_2) are collinear when $\begin{vmatrix} x_0 & y_0 & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0.$

Prob. 38. Write the conditions that all the points (x_1, y_1), (x_2, y_2), ... (x_n, y_n) shall be collinear in the form of a matrix.

Prob. 39. Obtain the equation of a right line through (x_1, y_1) and (x_2, y_2) in the form of a determinant.

Prob. 40. Show that the equation $\begin{vmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{vmatrix} = 0$

represents a plane through (x_1, y_1, z_1), (x_2, y_2, z_2), and (x_3, y_3, z_3).

ART. 30. THE MULTIPLICATION THEOREM.

Let the two homogeneous linear equations

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 = 0, \\ a_{21}x_1 + a_{22}x_2 = 0, \end{aligned}\right\} \qquad (1)$$

be subjected to linear transformation by substituting

$$\left.\begin{aligned} x_1 = b_{11}u_1 + b_{21}u_2, \\ x_2 = b_{12}u_1 + b_{22}u_2. \end{aligned}\right\} \qquad (2)$$

The result of such transformation is

$$\left.\begin{aligned}(a_{11}b_{11}+a_{12}b_{12})u_1+(a_{11}b_{21}+a_{12}b_{22})u_2&=0,\\(a_{21}b_{11}+a_{22}b_{12})u_1+(a_{21}b_{21}+a_{22}b_{22})u_2&=0.\end{aligned}\right\}\tag{3}$$

The vanishing of the determinant

$$\begin{vmatrix}a_{11}b_{11}+a_{12}b_{12} & a_{11}b_{21}+a_{12}b_{22}\\ a_{21}b_{11}+a_{22}b_{12} & a_{21}b_{21}+a_{22}b_{22}\end{vmatrix}\cdot\tag{4}$$

is the condition that the equations (3) may be consistent; that is, the condition that they may have solutions other than $u_1=0=u_2$ (Art. 27). Now the equations (3) may be consistent because of the consistency of the equations (1), in which case the determinant

$$\begin{vmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{vmatrix}\tag{5}$$

vanishes. Or, this condition failing, and the equations (1) thus having no solution other than $x_1=0=x_2$, the equations (3) will still be consistent if the equations (2) are so; that is, if the determinant

$$\begin{vmatrix}b_{11} & b_{12}\\ b_{21} & b_{22}\end{vmatrix}\tag{6}$$

vanishes. The vanishing of either of the determinants (5) or (6), therefore, causes the determinant (4) to vanish. It follows that (5) and (6) are factors of (4); and since their product and the determinant (4) are of the same degree with respect to the coefficients $a_{11},\ldots,b_{11},\ldots$, they are the only factors. Hence,

$$\begin{vmatrix}a_{11}a_{12}\\ a_{21}a_{22}\end{vmatrix}\cdot\begin{vmatrix}b_{11}b_{12}\\ b_{21}b_{22}\end{vmatrix}=\begin{vmatrix}a_{11}b_{11}+a_{12}b_{12} & a_{11}b_{21}+a_{12}b_{22}\\ a_{21}b_{11}+a_{22}b_{12} & a_{21}b_{21}+a_{22}b_{22}\end{vmatrix}\cdot\tag{7}$$

The above method is equally applicable to the formation of the product of any two determinants of the same order. Hence results the following general formula:

$$|\, a_{11}\, a_{22}\ldots a_{nn}\,|\cdot|\, b_{11}\, b_{22}\ldots b_{nn}\,|=$$

$$\begin{vmatrix}a_{11}b_{11}+\ldots+a_{1n}b_{1n} & a_{11}b_{21}+\ldots+a_{1n}b_{2n} & \ldots & a_{11}b_{n1}+\ldots+a_{1n}b_{nn}\\ a_{21}b_{11}+\ldots+a_{2n}b_{1n} & a_{21}b_{21}+\ldots+a_{2n}b_{2n} & \ldots & a_{21}b_{n1}+\ldots+a_{2n}b_{nn}\\ \ldots & \ldots & \ldots & \ldots\\ a_{n1}b_{11}+\ldots+a_{nn}b_{1n} & a_{n1}b_{21}+\ldots+a_{nn}b_{2n} & \ldots & a_{n1}b_{n1}+\ldots+a_{nn}b_{nn}\end{vmatrix}.\tag{8}$$

The process indicated by this formula may be described as follows: *

To form the determinant $|p_{1,n}|$, which is the product of two determinants $|a_{1,n}|$ and $|b_{1,n}|$, first connect by plus signs the elements in the rows of both $|a_{1,n}|$ and $|b_{1,n}|$. Then place the first row of $|a_{1,n}|$ upon each row of $|b_{1,n}|$ in turn and let each two elements as they touch become products. This is the first row of $|p_{1,n}|$. Perform the same operation upon $|b_{1,n}|$ with the second row of $|a_{1,n}|$ to obtain the second row of $|p_{1,n}|$; and again with the third row of $|a_{1,n}|$ to obtain the third row of $|p_{1,n}|$; etc.

Any element of this product is

$$p_{ks} = a_{k_1}b_{s_1} + a_{k_2}b_{s_2} + \dots + a_{kn}b_{sn}. \tag{9}$$

When the two determinants to be multiplied together are of different orders the one of lower order should be expressed as a determinant of the same order as the other (Art. 22), after which the above rule is applicable.

The product of two determinants may be formed by columns, instead of by rows as above. In this case the result is obtained in a different form. Thus the product of the determinants (5) and (6) by columns is

$$\begin{vmatrix} a_{11}b_{11} + a_{21}b_{21} & a_{12}b_{11} + a_{22}b_{12} \\ a_{11}b_{12} + a_{21}b_{22} & a_{12}b_{12} + a_{22}b_{22} \end{vmatrix}.$$

Prob. 41. Form the following products:

$$(1)\ \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} \cdot \begin{vmatrix} a & g \\ g & c \end{vmatrix}; \qquad (2)\ \begin{vmatrix} b & f \\ f & c \end{vmatrix} \cdot \begin{vmatrix} a & g \\ g & c \end{vmatrix} \cdot \begin{vmatrix} a & h \\ h & b \end{vmatrix};$$

$$(3)\ \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \cdot \begin{vmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{vmatrix}; \qquad (4)\ \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \cdot \begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix}.$$

Prob. 42. Generalize the last example (see Prob. 22, Art. 18).

Prob. 43. By forming the product

$$\begin{vmatrix} a+b\sqrt{-1} & -c+d\sqrt{-1} \\ c+d\sqrt{-1} & a-b\sqrt{-1} \end{vmatrix} \cdot \begin{vmatrix} j+k\sqrt{-1} & -l+m\sqrt{-1} \\ l+m\sqrt{-1} & j-k\sqrt{-1} \end{vmatrix},$$

* Carr's Synopsis of Pure Mathematics, London, 1886, Article 570.

show that the product of two numbers, each the sum of four squares, is itself the sum of four squares.

ART. 31. PRODUCT OF TWO ARRAYS.

The process explained in the preceding article may be applied to form what is conventionally termed the product of two rectangular arrays. It will appear, however, that multiplying two such arrays together by columns leads to a result radically different from that obtained when the product is formed by rows.

Let the two rectangular arrays be

$$\begin{vmatrix} a_{11}a_{12}a_{13} \\ a_{21}a_{22}a_{23} \end{vmatrix} \text{ and } \begin{vmatrix} b_{11}b_{12}b_{13} \\ b_{21}b_{22}b_{23} \end{vmatrix}.$$

The product of these by columns is

$$\Delta \equiv \begin{vmatrix} a_{11}b_{11} + a_{21}b_{21} & a_{12}b_{11} + a_{22}b_{21} & a_{13}b_{11} + a_{23}b_{21} \\ a_{11}b_{12} + a_{21}b_{22} & a_{12}b_{12} + a_{22}b_{22} & a_{13}b_{12} + a_{23}b_{22} \\ a_{11}b_{13} + a_{21}b_{23} & a_{12}b_{13} + a_{22}b_{23} & a_{13}b_{13} + a_{23}b_{23} \end{vmatrix}.$$

The determinant Δ is plainly equal to zero, being the product of two determinants formed by adding a row of zeros to one of the given rectangular arrays and a row of elements chosen arbitrarily to the other.

In general, the product by columns of two rectangular arrays having m rows and n columns, m being less than n, is a determinant of the n^{th} order whose value is zero.

Multiplying together the above rectangular arrays by rows, the result is

$$\Delta' \equiv \begin{vmatrix} a_{11}b_{11} + a_{12}b_{12} + a_{13}b_{13} & a_{11}b_{21} + a_{12}b_{22} + a_{13}b_{23} \\ a_{21}b_{11} + a_{22}b_{12} + a_{23}b_{13} & a_{21}b_{21} + a_{22}b_{22} + a_{23}b_{23} \end{vmatrix}$$

$$= \begin{vmatrix} a_{12}a_{13} \\ a_{22}a_{23} \end{vmatrix} \cdot \begin{vmatrix} b_{12}b_{13} \\ b_{22}b_{23} \end{vmatrix} + \begin{vmatrix} a_{11}a_{13} \\ a_{21}a_{23} \end{vmatrix} \cdot \begin{vmatrix} b_{11}b_{13} \\ b_{21}b_{23} \end{vmatrix} + \begin{vmatrix} a_{11}a_{12} \\ a_{21}a_{22} \end{vmatrix} \cdot \begin{vmatrix} b_{11}b_{12} \\ b_{21}b_{22} \end{vmatrix}.$$

In the same manner it may be shown that the product by rows of two rectangular arrays having m rows and n columns, m being less than n, is a determinant of the m^{th} order, which may be expressed as the sum of the $n!/m!(n-m)!$ determinants

formed from one of the arrays by deleting $(n - m)$ columns, each multiplied by the determinant formed by deleting the same columns from the other array.

ART. 32. RECIPROCAL DETERMINANTS.

The determinant formed by replacing each element of a given determinant by its co-factor is called the reciprocal of the given determinant.* Thus, the reciprocal of

$$\delta \equiv \begin{vmatrix} a_{11}a_{12}\ldots a_{1n} \\ a_{21}a_{22}\ldots a_{2n} \\ \ldots\ldots\ldots \\ a_{n1}a_{n2}\ldots a_{nn} \end{vmatrix} \quad \text{is} \quad \Delta \equiv \begin{vmatrix} A_{11}A_{12}\ldots A_{1n} \\ A_{21}A_{22}\ldots A_{2n} \\ \ldots\ldots\ldots \\ A_{n1}A_{n2}\ldots A_{nn} \end{vmatrix}.$$

The product of these two determinants is

$$\delta . \Delta = \begin{vmatrix} a_{11}A_{11}+\ldots+a_{1n}A_{1n} & a_{11}A_{21}+\ldots+a_{1n}A_{2n} & \ldots & a_{11}A_{n1}+\ldots+a_{1n}A_{nn} \\ a_{21}A_{11}+\ldots+a_{2n}A_{1n} & a_{21}A_{21}+\ldots+a_{2n}A_{2n} & \ldots & a_{21}A_{n1}+\ldots+a_{2n}A_{nn} \\ \ldots & \ldots & \ldots & \ldots \\ a_{n1}A_{11}+\ldots+a_{nn}A_{1n} & a_{n1}A_{21}+\ldots+a_{nn}A_{2n} & \ldots & a_{n1}A_{n1}+\ldots+a_{nn}A_{nn} \end{vmatrix}.$$

Each element on the principal diagonal of this product is equal to δ (Art. 18), while all the other elements vanish (Art. 19). Hence,

$$\delta . \Delta = \begin{vmatrix} \delta & 0 & \ldots & 0^{(n)} \\ 0 & \delta & \ldots & 0 \\ \ldots & & & \\ 0_n & 0 & \ldots & \delta \end{vmatrix} = \delta^n, \quad \text{or} \quad \Delta = \delta^{n-1}.$$

That is, the reciprocal of a determinant of the n^{th} order is equal to its $(n - 1)^{\text{th}}$ power.

ART. 33. LINEAR TRANSFORMATIONS.

Let it be required to transform the system

$$a_{h1}x_1 + a_{h2}x_2 + \ldots + a_{hn}x_n = 0 \quad [h = 1, 2, \ldots n] \qquad (1)$$

into a new system with the variables $u_1, u_2, \ldots u_n$ by means of the linear substitutions

$$x_i = b_{1i}u_1 + b_{2i}u_2 + \ldots + b_{ni}u_n. \quad [i = 1, 2, \ldots n] \qquad (2)$$

* [illegible] adjoint

Making the required transformation, as has been done for the case of two variables in Art. 30, the resulting system is

$$p_{j1}u_1+p_{j2}u_2+\ldots+p_{jn}u_n=0, \quad [j=1, 2, \ldots n] \tag{3}$$

in which $\quad p_{ks}=a_{k1}b_{s1}+a_{k2}b_{s2}+\ldots+a_{kn}b_{sn}.\quad$ (Eq. 9, Art. 30.)

The determinants of the systems (1), (2), and (3) are thus connected by the relation

$$|\,p_{1,n}\,|=|\,a_{1,n}\,|\cdot|\,b_{1,n}\,|. \tag{4}$$

The determinant $|\,b_{1,n}\,|$, whose elements are the coefficients in the equations (2), is called the modulus of transformation; and the relation expressed by equation (4) may be stated as follows:

If a system of n homogeneous linear equations in n variables be subjected to linear transformation, the eliminant of the transformed equations will be the eliminant of the given equations multiplied by the modulus of transformation. A transformation whose modulus is unity is said to be unimodular.

Prob. 44. Show that the following transformations are unimodular:

(1) $\quad x=x'+y'+2z', \quad y=x'+y'+z', \quad z=y'+z';$

(2) $\quad x=x'\cos\alpha-y'\sin\alpha, \quad y=x'\sin\alpha+y'\cos\alpha.$

Art. 34. Quantics; Invariants and Covariants.

A homogeneous function of any number of variables is called a quantic.

A quantic is binary, ternary, ... n-ary according as it contains two, three, ... n variables; and is specifically known as a quadric, cubic, ... m-ic according as it is of the second, third, ... mth degree. Thus, the function

$$q\equiv a_{11}x_1^2+a_{22}x_2^2+a_{33}x_3^2+2a_{23}x_2x_3+2a_{31}x_3x_1+2a_{12}x_1x_2$$

is a ternary quadric.

A covariant is a quantic derived from another quantic in such manner that when both are transformed by the same linear substitutions the resulting quantics are still connected by the same process of derivation.

An invariant is a function of the coefficients of a quantic which is not effected by linear transformation of the quantic, except that it is multiplied by a power of the modulus.

It is obvious that every invariant of a covariant is an invariant of the original quantic.

Art. 35. The Discriminant.

The discriminant of a quantic is the eliminant of its first derivatives.

Thus, the binary quadric

$$\phi \equiv a_{11}x_1^2 + a_{22}x_2^2 + 2a_{12}x_1x_2 = 0$$

gives

$$\frac{1}{2}\frac{\partial\phi}{\partial x_1} = a_{11}x_1 + a_{12}x_2 = 0, \quad \frac{1}{2}\frac{\partial\phi}{\partial x_2} = a_{12}x_1 + a_{22}x_2 = 0.$$

Hence, writing $a_{12} \equiv a_{21}$, the discriminant is

$$\delta \equiv \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}.$$

If ϕ be transformed to ϕ' by the substitutions

$$x_1 = b_{11}u_1 + b_{12}u_2, \quad x_2 = b_{21}u_1 + b_{22}u_2,$$

and the discriminant δ' of ϕ' be formed, then

$$\delta' \equiv \begin{vmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{vmatrix} . \delta.$$

Thus δ is an invariant of ϕ.

The notation for the n-ary quadric is

$$q \equiv \Sigma\Sigma a_{ks}x_kx_s, \quad \begin{bmatrix} k = 1, 2, \dots n \\ s = 1, 2, \dots n \end{bmatrix},$$

in which a_{kk} is the coefficient of x_k^2, while that of x_kx_s is a_{ks}. The semi-derivatives of q are, upon writing $a_{ks} \equiv a_{sk}$,

$$q_i \equiv \frac{1}{2}\frac{\partial q}{\partial x_i} = a_{1i}x_1 + a_{2i}x_2 + \dots + a_{ni}x_n. \quad [i = 1, 2, \dots n]$$

The discriminant is therefore the symmetrical determinant

$$\delta \equiv \begin{vmatrix} a_{11} & \dots & a_{1n} \\ \dots & \dots & \dots \\ a_{n1} & \dots & a_{nn} \end{vmatrix}.$$

It will be shown in Art. 42 that the discriminant of every quadric is an invariant. This also follows from Art. 30.

Art. 36. Composite Quadrics.

Whenever a quadric is resolvable into linear factors its discriminant vanishes.

To prove this, let

$$\Sigma\Sigma a_{ks}x_kx_s=(b_1x_1+\ldots+b_nx_n)(c_1x_1+\ldots+c_nx_n).$$

Equating coefficients gives

$$a_{kk}=b_kc_k,\quad 2a_{ks}=b_kc_s+b_sc_k.\quad \begin{bmatrix}k=1,2,\ldots n\\ s=1,2,\ldots n\end{bmatrix}\tag{1}$$

Now, the product of the two rectangular arrays

$$\begin{vmatrix} b_1b_2\ldots b_n\\ c_1c_2\ldots c_n\end{vmatrix}\cdot\begin{vmatrix} c_1c_2\ldots c_n\\ b_1b_2\ldots b_n\end{vmatrix}$$

is the determinant (Art. 31)

$$\begin{vmatrix} 2b_1c_1 & b_2c_1+b_1c_2 & \ldots b_nc_1+b_1c_n\\ b_1c_2+b_2c_1 & 2b_2c_2 & \ldots b_nc_2+b_2c_n\\ \cdots & \cdots & \cdots\\ b_1c_n+b_nc_1 & b_2c_n+b_nc_2\ldots & 2b_nc_n\end{vmatrix}=0$$

Hence, substituting from the equations (1),

$$\delta\equiv\begin{vmatrix} a_{11}\ldots a_{1n}\\ \cdots\cdots\\ a_{n1}\ldots a_{nn}\end{vmatrix}=0.$$

Art. 37. Discriminant of Binary Quantic an Invariant.

When a binary quantic contains a square factor its discriminant vanishes.

For, any such quantic is of the form

$$q\equiv(a_1x_1+a_2x_2)^2f(x_1,y_1),$$

and each of the derivatives $\dfrac{\partial q}{\partial x_1}$ and $\dfrac{\partial q}{\partial x_2}$ contains $(a_1x_1+a_2x_2)$ as a factor. Their eliminant, which is the discriminant of q, must therefore vanish.

A square factor remains such after linear transformation. If, therefore, the discriminant vanishes, that of the transformed function also vanishes and thus contains the first as a factor. Hence, the discriminant of a binary quantic is an invariant.

Art. 38. The Jacobian.

Let $y_1, y_2, \ldots y_n$ be n functions, each of the n independent variables $x_1, x_2, \ldots x_n$. Then the determinant

$$J \equiv \begin{vmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_2}{\partial x_1} & \ldots & \frac{\partial y_n}{\partial x_1} \\ \frac{\partial y_1}{\partial x_2} & \frac{\partial y_2}{\partial x_2} & \ldots & \frac{\partial y_n}{\partial x_2} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial y_1}{\partial x_n} & \frac{\partial y_2}{\partial x_n} & \ldots & \frac{\partial y_n}{\partial x_n^2} \end{vmatrix}$$

is called the Jacobian of the given functions.

The notation

$$J \equiv \frac{d(y_1, y_2, \ldots y_n)}{d(x_1, x_2, \ldots x_n)}$$

is in common use, being suggested by the close analogy between the Jacobian and the ordinary differential coefficient.

When the functions are linear, thus:

$$y_i = a_{1i}x_1 + a_{2i}x_2 + \ldots + a_{ni}x_n \quad [i = 1, 2, \ldots n]$$

it follows from the above definition that the Jacobian is the determinant of the coefficients. That is,

$$J \equiv | a_{11}a_{22} \ldots a_{nn} | .$$

When the functions are not independent; that is, when $F(y_1, y_2, \ldots y_n) = 0$, the Jacobian vanishes.

For, differentiating this function with respect to each of the variables $x_1, x_2, \ldots x_n$ gives the consistent system

$$\frac{\partial F}{\partial y_1}\frac{\partial y_1}{\partial x_i} + \frac{\partial F}{\partial y_2}\frac{\partial y_2}{\partial x_i} + \ldots + \frac{\partial F}{\partial y_n}\frac{\partial y_n}{\partial x_i} = 0; \quad [i = 1, 2, \ldots, n]$$

and eliminating $\frac{\partial F}{\partial y_1}, \ldots \frac{\partial F}{\partial y_n}$ from these equations there results

$$\left| \frac{\partial y_1}{\partial x_1} \frac{\partial y_2}{\partial x_2} \cdots \frac{\partial y_n}{\partial x_n} \right| \equiv J = 0.$$

Art. 39. Jacobian of Indirect Functions.

When $y_1, y_2, \ldots y_n$ are each functions of $\zeta_1, \zeta_2, \ldots \zeta_n$, these being functions each of $x_1, x_2, \ldots x_2$, then

$$\frac{d(y_1, y_2, \ldots y_n)}{d(x_1 x_2, \ldots x_n)} = \frac{d(y_1, y_2, \ldots y_n)}{d(\zeta_1, \zeta_2, \ldots \zeta_n)} \cdot \frac{d(\zeta_1, \zeta_2, \ldots \zeta_n)}{d(x_1, x_2, \ldots x_n)}.$$

This may be demonstrated by writing out each of the Jacobians in the second member in determinant form, changing columns into rows in the first, multiplying the two together by rows, and interpreting the result by means of the relation

$$\frac{dy_i}{dx_k} = \frac{\partial y_i}{\partial \zeta_1}\frac{\partial \zeta_1}{\partial x_k} + \frac{\partial y_i}{\partial \zeta_2}\frac{\partial \zeta_2}{\partial x_k} + \ldots + \frac{\partial y_i}{\partial \zeta_n}\frac{\partial \zeta_n}{\partial x_k}.$$

Thus, for $n=2$,

$$\frac{d(y_1, y_2)}{d(\zeta_1, \zeta_2)} \cdot \frac{d(\zeta_1, \zeta_2)}{d(x_1, x_2)} = \begin{vmatrix} \frac{\partial y_1}{\partial \zeta_1} & \frac{\partial y_1}{\partial \zeta_2} \\ \frac{\partial y_2}{\partial \zeta_1} & \frac{\partial y_2}{\partial \zeta_2} \end{vmatrix} \cdot \begin{vmatrix} \frac{\partial \zeta_1}{\partial x_1} & \frac{\partial \zeta_2}{\partial x_1} \\ \frac{\partial \zeta_1}{\partial x_2} & \frac{\partial \zeta_2}{\partial x_2} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{\partial y_1}{\partial \zeta_1}\;\frac{\partial \zeta_1}{\partial x_1} + \frac{\partial y_1}{\partial \zeta_2}\;\frac{\partial \zeta_2}{\partial x_1} & \frac{\partial y_1}{\partial \zeta_1}\;\frac{\partial \zeta_1}{\partial x_2} + \frac{\partial y_1}{\partial \zeta_2}\;\frac{\partial \zeta_2}{\partial x_2} \\ \frac{\partial y_2}{\partial \zeta_1}\;\frac{\partial \zeta_1}{\partial x_1} + \frac{\partial y_2}{\partial \zeta_2}\;\frac{\partial \zeta_2}{\partial x_1} & \frac{\partial y_2}{\partial \zeta_1}\;\frac{\partial \zeta_1}{\partial x_2} + \frac{\partial y_2}{\partial \zeta_2}\;\frac{\partial \zeta_2}{\partial x_2} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} \end{vmatrix} = \frac{d(y_1, y_2)}{d(x_1, x_2)}.$$

It may be shown in precisely similar manner that, when the functions $y_1, y_2, \ldots y_n$ are independent,

$$\frac{d(y_1, y_2, \ldots y_n)}{d(x_1, x_2, \ldots x_n)} \cdot \frac{d(x_1, x_2, \ldots x_n)}{d(y_1, y_2, \ldots y_n)} = 1.$$

ART. 40. THE JACOBIAN A COVARIANT.

When, in the preceding article, the functions $\zeta_1\, \zeta_2, \ldots \zeta_n$ are linear, thus:

$$\zeta_i = a_{1i}x_1 + a_{2i}x_2 + \ldots + a_{ni}x_n; \quad [i=1, 2, \ldots n]$$

then (Art. 38)

$$\frac{d(y_1, y_2, \ldots y_n)}{d(x_1, x_2, \ldots x_n)} = |\, a_{11}a_{22} \ldots a_{nn}| \frac{d(y_1, y_2, \ldots y_n)}{d(\zeta_1, \zeta_2, \ldots \zeta_n)}.$$

Hence, if a set of functions be subjected to linear transformation, the Jacobian of the transformed functions is equal to that of the given functions multiplied by the modulus of the transformation. That is, the Jacobian is a covariant of the set of functions from which it is derived; unless these functions are linear, in which case it is an invariant.

ART. 41. JACOBIAN OF IMPLICIT FUNCTIONS.

When the functions are implicit, thus:

$$\phi_i(y_1, y_2, \ldots y_n, x_1, x_2, \ldots x_n) = 0; \quad [i=1, 2, \ldots n]$$

then

$$\frac{d(y_1, y_2, \ldots y_n)}{d(x_1, x_2, \ldots x_n)} = (-1)^n \frac{\dfrac{d(\phi_1, \phi_2, \ldots \phi_n)}{d(x_1, x_2, \ldots x_n)}}{\dfrac{d(\phi_1, \phi_2, \ldots \phi_n)}{d(y_1, y_2, \ldots y_n)}}.$$

To prove this, write the above Jacobians in determinant form, change columns into rows in the first member, and clear of fractions. This gives, representing by P the resulting product in the first member,

$$P \equiv \left| \frac{\partial \phi_i}{\partial y_1} \frac{\partial y_1}{\partial x_k} + \ldots + \frac{\partial \phi_i}{\partial y_n} \frac{\partial y_n}{\partial x_k} \right|.$$

Now, the total derivative of ϕ_i with respect to x_k is

$$\frac{d\phi_i}{dx_k}=\frac{\partial\phi_i}{\partial y_1}\ \frac{\partial y_1}{\partial x_k}+\ldots+\frac{\partial\phi_i}{\partial y_n}\ \frac{\partial y_n}{\partial x_k}$$
$$+\frac{\partial\phi_i}{\partial x_1}\ \frac{\partial x_1}{\partial x_k}+\ldots+\frac{\partial\phi_i}{\partial x_k}\ \frac{\partial x_k}{\partial x_k}+\ldots+\frac{\partial\phi_i}{\partial x_n}\ \frac{\partial x_n}{\partial x_k}=0.$$

But, since $x_1, x_2, \ldots x_n$ are independent, this becomes

$$\frac{\partial\phi_i}{\partial y_1}\ \frac{y_1}{\partial x_k}+\ldots+\frac{\partial\phi_i}{\partial y_n}\ \frac{\partial y_n}{\partial x_k}=-\frac{\partial\phi_i}{\partial x_k}.$$

The above product thus becomes

$$P=\left|-\frac{\partial\phi_i}{\partial x_k}\right|=(-1)^n\left|\frac{\partial\phi_i}{\partial x_k}\right|=(-1)^n\frac{d(\phi_1,\ \phi_2, \ldots \phi_n)}{d(x_1, x_2, \ldots x_n)},$$

which is the required proof.

Art. 42. The Hessian.

The Jacobian of the first differential coefficients of a function of n variables is called the Hessian of the function. Thus, the Hessian of $\phi(x_1, x_2, \ldots x_n)$ is

$$H(\phi)\equiv\frac{d\left(\dfrac{\partial\phi}{\partial x_1},\ \dfrac{\partial\phi}{\partial x_2},\ldots\dfrac{\partial\phi}{\partial x_n}\right)}{d(x_1,\quad x_2,\ \ldots x_n)}=\begin{vmatrix}\dfrac{\partial^2\phi}{\partial x_1{}^2} & \cdots & \dfrac{\partial^2\phi}{\partial x_n\partial x_1}\\ \cdots & \cdots & \cdots\\ \dfrac{\partial^2\phi}{\partial x_1\partial x_n} & \cdots & \dfrac{\partial^2\phi}{\partial x_n{}^2}\end{vmatrix}.$$

Since

$$\frac{\partial^2\phi}{\partial x_k\partial x_s}=\frac{\partial^2\phi}{\partial x_s\partial x_k},$$

the Hessian is a symmetric determinant.

The Hessian of a quadric differs from the discriminant only by a numerical factor.

Let the function ϕ be transformed into ϕ' by the linear substitutions

$$x_i=a_{1i}u_1+a_{2i}u_2+\ldots+a_{ni}\,x_n.\quad [i=1, 2, \ldots n]$$

Then (Art. 40)

$$H(\phi') \equiv \frac{d\left(\frac{\partial\phi'}{\partial u_1}, \frac{\partial\phi'}{\partial u_2}, \dots \frac{\partial\phi'}{\partial u_k}\right)}{d(u_1, u_2, \dots u_n)} = |\,a_{1,n}\,| \frac{d\left(\frac{\partial\phi'}{\partial u_1}, \frac{\partial\phi'}{\partial u_2}, \dots \frac{\partial\phi'}{\partial u_n}\right)}{d(x_1, x_2, \dots x_n)}.$$

But $\frac{\partial^2\phi'}{\partial x_s \partial u_k} = \frac{\partial^2\phi}{\partial u_k \partial x_s}$, and the above equation may therefore be written

$$H(\phi') = |\,a_{1,n}\,| \frac{d\left(\frac{\partial\phi}{\partial x_1}, \frac{\partial\phi}{\partial x_2}, \dots \frac{\partial\phi}{\partial x_n}\right)}{d(u_1, u_2, \dots u_n)}$$

$$= |\,a_{1,n}\,|^2 \frac{d\left(\frac{\partial\phi}{\partial x_1}, \frac{\partial\phi}{\partial x_2}, \dots \frac{\partial\phi}{\partial x_n}\right)}{d(x_1, x_2, \dots x_n)}$$

$$= |\,a_{1,n}\,|^2 H(\phi).$$

That is, if a function be subjected to linear transformation, the Hessian of the transformed function will equal that of the given function multiplied by the square of the modulus of the transformation. It follows that the Hessian is a covariant of the function from which it is derived; unless this function is a quadric, in which case it is an invariant (Art. 35).

Prob. 45. Tell whether or not the following quadrics are prime:

(1) $5x^2 - 9y^2 - 5z^2 + 18yz + 24zx - 12xy$;

(2) $x^2 + z^2 + yz - 2zx + xy$; (3) $9y^2 + 15yz - 6zx + 8xy$.

Prob. 46. Find the values of λ in order that each of the following quadrics may be composite:

(1) $\lambda xy + 5zx + 3yz + 2z^2$;

(2) $2x^2 - 3\lambda y^2 - 12z^2 + 17yz + \lambda zx - xy$;

(3) $4x^2 + 3z^2 - 5yz + 3zx + 2xy + \lambda(x^2 + 3y^2 - yz + 5zx + 3xy)$.

Prob. 47. Find the Jacobian of the functions

$$y_1 = 1 - x_1, \quad y_2 = x_1(1 - x_2), \quad y_3 = x_1x_2(1 - x_3),$$
$$\dots y_n = x_1x_2 \dots x_{n-1}(1 - x_n).$$

Prob. 48. Show, by means of their Jacobian, that the functions

$$y_1=(x-x_2)(x_2+x_3), \quad y_2=(x_1+x_2)(x_2-x_3), \quad y_3=x_2(x_3-x_1)$$

are not independent.

Prob. 49. Find $\dfrac{d(x, y)}{d(u, v)}$; having given $x=\rho\cos\theta$ and $y=\rho\sin\theta$, in which $\rho=au$, $\theta=bv$.

Prob. 50. Given $\dfrac{x\cos u}{v\cos y}=0$, $\dfrac{u\sin x}{y\sin v}=0$; to find $\dfrac{d(u, v)}{d(x, y)}$.

Prob. 51. Obtain the Hessian and the discriminant of: (*a*) the binary cubic; (*b*) the quaternary quadric; (*c*) the binary quartic.

Prob. 52. Find the Hessian of

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy,$$

and also that of the same function transformed by the substitutions.

$$x=l_1x'+m_1y_1+n_1z', \quad y=l_2x'+m_2y'+n_2z', \quad z=l_3x'+m_3y'+n_3z'.$$

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